GROUP PROBLEMS:

1. \[ \int_{1}^{\infty} \frac{\ln x}{x^3} \, dx \] Divergent

2. \[ \int_{-\infty}^{\infty} \frac{x}{1 + x^2} \, dx \] Divergent

3. \[ \int_{-1}^{1} \frac{x + 1}{3\sqrt[3]{x^4}} \, dx \] Divergent

4. \[ \int_{0}^{1} \frac{t^2 + 1}{t^2 - 1} \, dt \] Divergent

5. \[ \int_{-\infty}^{\infty} \frac{x + 1}{(x^2 + 2x)^2} \, dx \] then \[ x \sin x \leq x \]

Comparison Theorem:

6. \[ \int_{0}^{\pi} \frac{2 + e^{-x}}{x} \, dx \]
\[ \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx \quad u = \ln x \quad du = \frac{1}{x} \, dx \quad \frac{1}{2}x^{-2} \theta = \frac{-1}{2x^2} \]

\[ = \lim_{t \to \infty} \left[ \left. -\frac{\ln x}{2x^2} \right|_{1}^{t} - \int_{1}^{t} \frac{-1}{2x^3} \, dx \right] \]

\[ = \lim_{t \to \infty} \left[ \left. -\frac{\ln x}{2x^2} \right|_{1}^{t} - \frac{1}{4x^2} \right]_{1}^{t} \]

\[ = \lim_{t \to \infty} \left[ \left( \frac{-\ln t}{2t^2} - \frac{1}{4t^2} \right) - \left( \frac{\ln(1)}{2} - \frac{1}{4} \right) \right] \]

\[ = \lim_{t \to \infty} \frac{-\ln t}{2t^2} + \frac{1}{4} = \cdot \frac{-\infty}{\infty} + \frac{1}{4} \]

\[ = \lim_{t \to \infty} \frac{-\frac{1}{t}}{\frac{1}{4t^2}} + \frac{1}{4} = \lim_{t \to \infty} -\frac{1}{4t^2} + \frac{1}{4} = \sqrt{\frac{1}{4}} \]

Convergent.
2. \[ \int_{-\infty}^{\infty} \frac{x}{1+x^2} \, dx \quad \text{\textcolor{red}{\textbf{u = 1+x^2}}} \quad \text{\textcolor{red}{\textbf{du = 2xdx}}} \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{u} \, du \quad = \left[ \text{sketch} \right] \frac{1}{2} \int_{-\infty}^{1} \frac{1}{u} \, du + \frac{1}{2} \int_{1}^{\infty} \frac{1}{u} \, du \]

Consider:

\[ \int_{1}^{\infty} \frac{1}{u} \, du = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{u} \, du = \lim_{t \to \infty} [\ln|u|]_{1}^{t} \]

\[ = \lim_{t \to \infty} \ln|t| - \ln|1| = \lim_{t \to \infty} \ln|t| = \infty. \quad \text{Divergent.} \]
3. \( \int_{-1}^{1} \frac{x+1}{\sqrt[3]{x^4}} \, dx = \int_{-1}^{0} \frac{x+1}{\sqrt[3]{x^4}} \, dx + \int_{0}^{1} \frac{x+1}{\sqrt[3]{x^4}} \, dx \)

Consider

\[ \int_{0}^{1} \frac{x+1}{\sqrt[3]{x^4}} \, dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{x+1}{x^{4/3}} \, dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{x}{x^{4/3}} + \frac{1}{x^{4/3}} \, dx \]

\[ = \lim_{t \to 0^+} \int_{t}^{1} x^{-\frac{1}{3}} + x^{-\frac{4}{3}} \, dx \]

\[ = \lim_{t \to 0^+} \left[ \frac{3}{2} x^{\frac{2}{3}} + \frac{3}{2} x^{\frac{-1}{3}} \right]_{t}^{1} \]

\[ = \lim_{t \to 0^+} \left[ \left( \frac{3}{2} - 3 \right) - \left( \frac{3}{2} \sqrt[3]{t} - \frac{3}{2} \frac{1}{\sqrt[3]{t}} \right) \right] = -\frac{7}{3} + \lim_{t \to 0^+} \left( \frac{3}{3\sqrt[3]{t}} - \frac{3}{2} \frac{3}{\sqrt{3t}} \right) \]

\[ = -\frac{7}{3} + (\infty) - 0 = \infty \quad \text{Divergent} \]
4. \[ \int_0^1 \frac{t^2 + 1}{t^2 - 1} \, dt = \lim_{x \to 1} \int_0^x \frac{t^2 - 1 + 2}{t^2 - 1} \, dt \]

\[ = \lim_{x \to 1} \int_0^x 1 + \frac{2}{t^2 - 1} \, dt \]

\[ t^2 + 1 = (t-1)(t+1) \]

\[ \frac{2}{t^2 - 1} = \frac{A}{t+1} + \frac{B}{t-1} \]

\[ 2 = A(t-1) + B(t+1) \]

\[ t=1: \quad 2 = 2B, \quad B = 1 \]

\[ t=-1: \quad 2 = -2A, \quad A = -1 \]

\[ = \lim_{x \to 1} \left[ x - \ln|x+1| + \ln|x-1| - (\ln1 + \ln|1|) \right] \]

\[ = \lim_{x \to 1} \left[ \ln|x-1| - \ln2 + 1 \right] = -\infty \]

Divergent
\[ 5. \int_{1}^{\infty} \frac{x+1}{(x^2+2x)^2} \, dx \]

\[ u = x^2 + 2x \]

\[ du = 2x + 2 \, dx = 2(x+1) \, dx \]

\[ = \frac{1}{2} \int_{3}^{\infty} \frac{1}{u^2} \, du \]

\[ = \frac{1}{2} \left[ \lim_{t \to \infty} \frac{-1}{u} \right]_{3}^{t} \]

\[ = \frac{1}{2} \left[ \lim_{t \to \infty} \frac{-1}{t} + \frac{1}{3} \right] \]

\[ = \frac{1}{2} \left[ 0 + \frac{1}{3} \right] \]

\[ = \frac{1}{6} \]

convergent
6. \[ \int_0^{\pi/2} \frac{dx}{x \sin x} \]

Do this problem instead:

\[ \int_1^{\infty} \frac{2 + e^{-x}}{x} \, dx \]

\( e^{-x} > 0 \) for all real values of \( x \)

So \( \frac{2 + e^{-x}}{x} > \frac{2}{x} > \frac{1}{x} \)

And, by our Fact \( \int_1^{\infty} \frac{1}{x} \, dx \) is divergent.

Thus, by the Comparison Theorem, \( \int_1^{\infty} \frac{2 + e^{-x}}{x} \, dx \) is divergent, as well.