9.3

1. Solve the differential equation \( \frac{dy}{dx} = \frac{\sqrt{x}}{e^y} \)

\[
e^y \, dy = \sqrt{x} \, dx
\]

\[
\int e^y \, dy = \int \sqrt{x} \, dx
\]

\[
e^y = \frac{2}{3} x^{3/2} + C
\]

\[y = \ln \left( \frac{2}{3} x^{3/2} + C \right)\]

2. Solve the initial-value problem: \( \frac{dy}{dx} = \frac{y \cos x}{1+y^2}, \ y(0) = 1 \)

\[
\frac{dy}{dx} = \frac{y}{1+y^2} \cdot \cos x
\]

\[
\frac{1+y^2}{y} \, dy = \cos x \, dx
\]

\[
\left( \frac{1}{y} + \frac{y^2}{y} \right) \, dy = \cos x \, dx
\]

\[
\int \frac{1}{y} + y \, dy = \int \cos x \, dx
\]

\[\ln |y| + \frac{1}{2} y^2 = \sin x + C\]

Since \( y(0) = 1 \):

\[\ln 1 + \frac{1}{2} (1^2) = \sin(0) + C\]

\[0 + \frac{1}{2} = 0 + C\]

\[C = \frac{1}{2}\]

\[\ln |y| + \frac{1}{2} y^2 = \sin x + \frac{1}{2}\]
3. Solve the initial-value problem: \( xy' + y = y^2 \), \( y(1) = -1 \)

\[
x \left( \frac{dy}{dx} \right) = y^2 - y
\]

\[
\frac{dy}{dx} = \frac{y(y-1)}{x}
\]

\[
\int \frac{-1}{y} + \frac{1}{y-1} \, dy = \int \frac{1}{x} \, dx
\]

- \( \ln |y| + \ln |y-1| = \ln |x| + C \)
- \( \ln \left| \frac{y-1}{y} \right| = \ln |x| + C \)
- \( \frac{y-1}{y} = x \cdot e^C = Kx \)
- \( y-1 = Kxy \)
- \( y-Kxy = 1 \)
- \( y(1-Kx) = 1 \)
- \( y = \frac{1}{1-Kx} \)

Since \( y(1) = -1 \): \[
-1 = \frac{1}{1-K(1)}
\]
- \( -1 = \frac{1}{1-K} \)
- \( K-1 = 1 \)
- \( K = 2 \)

So, the solution to the IVP is \( y = \frac{1}{1-2x} \).
4. A tank contains 1000L of pure water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 5 liters per minute. Brine that contains 0.04 kg of salt per liter of water enters the tank at a rate of 10 liters per minute. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 liters per minute.

How much salt is in the tank after (a) t minutes? (b) one hour?

(a)
If \( y(t) \) is the amount of salt in kg after \( t \) minutes, then \( y(0) = 0 \), and the total amount of liquid in the tank remains constant at 1000 L.

\[
\frac{dy}{dt} = \left( 0.05 \text{ kg/L} \right) \left( 5 \text{ L/min} \right) + \left( 0.04 \text{ kg/L} \right) \left( 10 \text{ L/min} \right) - \left( \frac{y(t)}{1000} \text{ kg/L} \right) \left( 15 \text{ L/min} \right)
\]

\[
= 0.25 + 0.4 - 0.015 y \quad \text{kg/min}
\]

\[
= 0.65 - 0.015 y \quad \text{kg/min}
\]

\[
\frac{dy}{dt} = \frac{130 - 3y}{200} \quad \text{kg/min}
\]
\[ \frac{dy}{dt} = \frac{130 - 3y}{200} \quad \text{so} \quad \frac{1}{130 - 3y} \, dy = \frac{1}{200} \, dt \]

\[ \int \frac{1}{130 - 3y} \, dy = \int \frac{1}{200} \, dt \]

\[ -\frac{1}{3} \ln |130 - 3y| = \frac{t}{200} + C \]

\[ \ln |130 - 3y| = -\frac{3t}{200} + C \quad \text{let} \quad C = e^C \]

\[ 130 - 3y = e^{-\frac{3t}{200}} + C = e^{-\frac{3t}{200}} e^C = K e^{-\frac{3t}{200}} \]

\[ -3y = Ke^{-\frac{3t}{200}} - 130 \]

\[ y = Ke^{\frac{3t}{200}} + \frac{130}{3} \]

Since \( y(0) = 0 \) we have ***

\[ 0 = K(e^0) + \frac{130}{3} \]

\[ -\frac{130}{3} = K \]

Thus, the solution (the amount of salt in the tank after \( t \) minutes) is

\[ y = \frac{130}{3} \left( 1 - e^{-\frac{3t}{200}} \right) \text{ kg} \]

(for part (b), just let \( t = 0 \).)