Real Analysis – April 2011

Part A

1. Let $f \in L^1(\mathbb{R})$ and let $g(x) = \displaystyle\int_x^\infty f(t)\,dt$. Show that $g(x) \to 0$ as $x \to \infty$.

Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $x_n \to \infty$ as $n \to \infty$, and define $h_n(t) := \chi_{[x_n,\infty)} f(t)$. Now, given $\varepsilon > 0$, there is some N such that $\left|h_n(t)\right| = \left|\chi_{[x_n,\infty)} f(t)\right| = 0 < \varepsilon$. Thus, $\lim_{n\to\infty} h_n(t) = 0$. Also, notice that $\left|h_n(t)\right| \le \left|f(t)\right|$, so since $f \in L^1(\mathbb{R})$, by the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{x\to\infty} g(x) &= \lim_{n\to\infty} g(x_n) \\
&= \lim_{n\to\infty} \int_{x_n}^\infty f(t)\,dt \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{[x_n,\infty)} f(t)\,dt \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} h_n(t)\,dt \\
&= \int_{\mathbb{R}} \lim_{n\to\infty} h_n(t)\,dt \\
&= \int_{\mathbb{R}} 0\,dt \quad = \quad 0, \text{ as required.}
\end{aligned}$$

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2. Prove or disprove: If $f : [0,1] \to [0,1]$ is nondecreasing and continuous, then f is absolutely continuous.

False. Counterexample: Cantor function
Let C be the Cantor set. That is, let

$$\begin{aligned} C &= \left\{x \in [0,1] : x = \textstyle\sum_{n=1}^{\infty} a_n 3^{-n}, \text{ where each } a_n = 0 \text{ or } 2\right\} \\ &= \left\{x \in [0,1] \setminus \bigcup_{m=1}^{\infty} \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)\right\} \\ &= \left\{x \in [0,1] \setminus \text{“open middle thirds”}\right\}. \end{aligned}$$

Notice that $\mu(C) = 0$, since C is totally disconnected, where μ is the Lebesgue measure.
Define the Cantor function $f : [0,1] \to [0,1]$ as

$$f(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{2}\, 2^{-n}, & \text{if } x \in C \text{as defined above} \\ p\, 2^{-k}, \text{ for some } p, k \in \mathbb{Z}, & \text{if } x \in [0,1] \setminus C \end{cases}$$

so that if $x \in [0,1] \setminus C$, then $f(x)$ is equal to the value of f at the endpoints of the “open middle third” interval that x is in.
Notice: $f(0) = 0$ and $f(1) = 1$, (i.e. f goes from 0 to 1 as x goes from 0 to 1). Also, f is continuous everywhere.
Now, since $\mu(C) = 0$ and $f'(x) = 0$ for every $x \notin C$, f is differentiable almost everywhere with $f' \equiv 0$. Thus, f is nondecreasing, also. So f satisfies the hypotheses.
However,

$$\int_0^1 f'(x)\, dx = \int_0^1 0\, dx = 0 \neq 1 = f(1) - f(0).$$

So, by the contrapositive of the Fundamental Theorem of Calculus, f cannot be absolutely continuous. ∎

3. Let E be a closed Lebesgue measurable subset of $[0,1]$. Prove or disprove:
a. If E^c is dense, then E has measure 0.
b. If E has measure 0, then E^c is dense.

a. False. Counterexample:
Let $E^c = \mathbb{Q} \cap (0,1)$. Notice: E^c is dense in $[0,1]$ since the closure of E^c is $[0,1]$ and since $\mathbb{Q}$ is dense in $\mathbb{R}$. Also, notice that E is closed since E^c is open.
But, $0 \leq \mu(E^c) \leq \mu(\mathbb{Q}) = 0$, so, $\mu(E) = \mu([0,1]) - \mu(E^c) = 1 - 0 = 1$,if μ is the Lebesgue measure.
b. True. Proof by contrapositive:
If E^c is not dense, then there is some $\delta > 0$ such that $(y, y+\delta)$ is not in E^c, for some $y \in E^c$.
Thus, $(y, y+\delta) \in E$ and $\mu(E) \geq \mu\Big((y, y+\delta)\Big) = \delta > 0$. ∎

4. Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, and assume that $f' \in L^4([0,1])$ and that $f(0) = 0$. Find all values of α so that

$$\lim_{x\to 0^+} x^{-\alpha} f(x) = 0$$

for all such f.

Since f is absolutely continuous on $\mathbb{R}$, it is also absolutely continuous on $[0,x]$, for any $x \in \mathbb{R}$, but let's only consider $x \in [0,1]$. Also, f' exists almost everywhere and is integrable on $[0,x]$, and, by the Fundamental Theorem of Calculus, $\int_0^x f'(t)\,dt = f(x) - f(0) = f(x)$.

Also, note that $f' \in L^4([0,1])$ implies that $\left(\int_0^1 |f'(t)|^4\,dt\right)^{1/4} < \infty$.

Now,

$$\begin{aligned}
0 \le \left|\lim_{x\to 0^+} x^{-\alpha} f(x)\right| &= \lim_{x\to 0^+} \left|x^{-\alpha}\right| \cdot \left|f(x)\right| \\
&= \lim_{x\to 0^+} x^{-\alpha} \left|\int_0^x f'(t)\,dt\right| \\
&\le \lim_{x\to 0^+} x^{-\alpha} \int_0^x \left|1 \cdot f'(t)\right| dt \\
&= \lim_{x\to 0^+} x^{-\alpha} \left\|1 \cdot f'(t)\right\|_1 \qquad \text{on } [0,x] \\
&\le \lim_{x\to 0^+} x^{-\alpha} \|1\|_{4/3} \cdot \left\|f'(t)\right\|_4 \qquad \text{By Hölder's Inequality} \\
&= \lim_{x\to 0^+} x^{-\alpha} \left(\int_0^x |1|^{4/3}\,dt\right)^{3/4} \left(\int_0^x \left|f'(t)\right|^4 dt\right)^{1/4} \\
&= \lim_{x\to 0^+} x^{-\alpha} x^{\frac{3}{4}} \left(\int_0^x \left|f'(t)\right|^4 dt\right)^{1/4} \\
&= \lim_{x\to 0^+} x^{\frac{3}{4}-\alpha} \left(\int_{\mathbb{R}} \chi_{[0,x]} \left|f'(t)\right|^4 dt\right)^{1/4}
\end{aligned}$$

Now, let $\{x_n\}$ be a sequence of positive real numbers in $[0,1]$ such that $x_n \to 0$ as $n \to \infty$. Define $g_n(x) := \chi_{[0,x_n]} \left|f'(x)\right|^4$. Then, $g_n(x) \to 0$ as $n \to \infty$, and$g_n(x) \le \left|f'(x)\right|^4$, which is integrable, so by the Dominated Convergence Theorem, we have the following:

$$\lim_{x\to 0^+} \int_{\mathbb{R}} \chi_{[0,x]} \left|f'(t)\right|^4 dt = \lim_{n\to\infty} \int_{\mathbb{R}} g_n(t)\,dt = \int_{\mathbb{R}} \lim_{n\to\infty} g_n(t)\,dt = \int_{\mathbb{R}} 0\,dt = 0.$$

Thus, $\lim_{x\to 0^+} x^{-\alpha} f(x) \le \lim_{n\to\infty} x_n^{\frac{3}{4}-\alpha} \left(\int_{\mathbb{R}} g_n(t)\,dt\right)^{1/4} = 0$, provided $x_n^{\frac{3}{4}-\alpha} \not\to \infty$. So, since $x_n \to 0$, it must be that $\frac{3}{4} - \alpha \ge 0$.

Therefore, if $\alpha \le \frac{3}{4}$, then $\lim_{x\to 0^+} x^{-\alpha} f(x) = 0$, as required. ∎

Part B

5. Recall that, if (X, ρ) and (Y, σ) are metric spaces, a function $f : X \to Y$ is called Lipschitz continuous if there exists a constant $\lambda \geq 0$ such that

$$\sigma\big(f(x), f(y)\big) \leq \lambda \rho(x, y) \quad \text{for all} \quad x, y \in X.$$

Let X be a compact metric space, and let $Y = \mathbb{R}$. Show that the Lipschitz continuous functions are dense in the space of continuous functions with the uniform norm.

Stone-Weierstrass Theorem:
Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X, \mathbb{R})$, which contains a non-zero constant function. Then A is dense in $C(X, \mathbb{R})$ if, and only if, it separates points.

In our case: $A = Lip(X, \mathbb{R}) = \{f : X \to \mathbb{R} | f \text{ is Lipschitz continuous}\}$.

We will need to show the following:

1. $Lip(X, \mathbb{R})$ is a subalgebra of $C(X, \mathbb{R})$:
   (a) $Lip(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$.
   (b) $Lip(X, \mathbb{R})$ is a linear space.
   (c) $Lip(X, \mathbb{R})$ is closed under multiplication.
2. There is a non-zero constant function in $Lip(X, \mathbb{R})$.
3. $Lip(X, \mathbb{R})$ separates points.
4. X is Hausdorff.

Now, let's get to work...

1. (a) Let $x, y \in X$ and $f \in Lip(X, \mathbb{R})$, then $|f(x) - f(y)| \leq \lambda \rho(x, y)$.
   Now, given $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{\lambda}$, such that $\rho(x, x_0) < \delta$.
   Then $|f(x) - f(x_0)| \leq \lambda \rho(x, x_0) < \lambda \delta = \varepsilon$. Thus, $Lip(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$.

   (b) Let $a, b \in \mathbb{R}$, $f, g \in Lip(X, \mathbb{R})$, $x, y \in X$, then

$$\begin{aligned} \Big|\big(af + bg\big)(x) - \big(af + bg\big)(y)\Big| &= \Big|a\big[f(x) - f(y)\big] + b\big[g(x) - g(y)\big]\Big| \\ &\leq |a|\Big|f(x) - f(y)\Big| + |b|\Big|g(x) - g(y)\Big| \\ &\leq |a|\,\lambda_1\,\rho(x, y) \;+\; |b|\,\lambda_2\,\rho(x, y) \\ &= \Big(|a|\,\lambda_1 \;+\; |b|\,\lambda_2\Big)\,\rho(x, y), \\ &\qquad\qquad \text{where } \big(|a|\lambda_1 + |b|\lambda_2\big) \geq 0. \end{aligned}$$

   Thus, $(af + bg) \in Lip(X, \mathbb{R})$, so $Lip(X, \mathbb{R})$ is a linear space.

(c) Let $x, y \in X$, $f, g \in Lip(X, \mathbb{R})$. Since X is compact and $f, g \in C(X, \mathbb{R})$, there is some $M \in \mathbb{R}$ such that $\left|f(t)\right| \leq M$ and $\left|g(t)\right| \leq M$, for every $t \in X$. Then,

$$\begin{aligned}
\left|fg(x) - fg(y)\right| &= \left|f(x)g(x) - f(y)g(y)\right| \\
&= \left|f(x)g(x) - \left[f(y)g(x) - f(y)g(x)\right] - f(y)g(y)\right| \\
&= \left|g(x)\left[f(x) - f(y)\right] - f(y)\left[g(x) - g(y)\right]\right| \\
&\leq M\left|f(x) - f(y)\right| \; + \; M\left|g(x) - g(y)\right| \\
&\leq M\,\lambda_1\,\rho(x,y) \; + \; M\,\lambda_2\,\rho(x,y) \\
&= M(\lambda_1 + \lambda_2)\,\rho(x,y), \;\text{ where } M(\lambda_1 + \lambda_2) \geq 0.
\end{aligned}$$

Thus, $fg \in Lip(X, \mathbb{R})$, so $Lip(X, \mathbb{R})$ is closed under multiplication.

2. Let $x, y \in X$, and let $f = c \neq 0, c \in \mathbb{R}$.
   Consider $|f(x) - f(y)| = |c - c| = 0 = \lambda\rho(x, y)$, for $\lambda = 0 \leq 0$. Thus, $f \in Lip(X, \mathbb{R})$.

3. Let $x, y \in X$ such that $x \neq y$, and suppose $f(t) = \rho(t, q)$, for every $t \in X$, where $q \in X$. Then $\left|f(x) - f(y)\right| = \left|\rho(x, q) - \rho(y, q)\right| = \left|\rho(x, q) - \rho(q, y)\right| \leq \rho(x, y)$. Thus, $f \in Lip(X, \mathbb{R})$.
   Also, for $r \in X$ such that $r \neq q$, we have that $f(r) = \rho(r, q) > 0$, whereas $f(q) = \rho(q, q) = 0$. Thus, f separates points.

4. A topological space T is *Hausdorff* if for $x, y \in T$ such that $x \neq y$, there are disjoint open sets $U, V \subseteq T$ with $x \in U$ and $y \in V$.

   Now, given that X is compact and $\mathbb{R}$ is Hausdorff, let $f \in Lip(X, \mathbb{R})$ and $f(x) \neq f(y)$ in $\mathbb{R}$. Then, $f(x) \in U$ and $f(y) \in V$ for open $U, V \subseteq \mathbb{R}$ such that $U \cap V = \emptyset$.
   Since $Lip(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$ we know that $f^{-1}(U)$ and $f^{-1}(V)$ are both open in X.
   If $t \in f^{-1}(U) \cap f^{-1}(V)$, then $t \in f^{-1}(U)$ and $t \in f^{-1}(V)$, which implies that $f(t) \in U$ and $f(t) \in V$, but $U \cap V = \emptyset$. Thus, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.
   Therefore, X is Hausdorff.

Thus, by the Stone-Weierstrass Theorem, the Lipschitz continuous functions are dense in the space of continuous functions with the uniform norm, as required. ■

6. Use an appropriate Fourier series to compute $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

Consider a function $f : \mathbb{R} \to \mathbb{R}$ which is periodic with period one and which satisfies

$$f(x) \;=\; \chi_{(0,1/2)}(x) - \chi_{-1/2,0)}(x) \quad \text{for} \quad |x| < \frac{1}{2}$$

The Fourier series for f is $\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}$, where

$$\begin{aligned}
c_n &= \int_{\mathbb{R}} f(x)e^{-2\pi inx}\,dx \\
&= \int_0^{\frac{1}{2}} e^{-2\pi inx}\,dx \;-\; \int_{-\frac{1}{2}}^{0} e^{-2\pi inx}\,dx \\
c_n &= \left[\frac{e^{-2\pi inx}}{-2\pi in}\right]_0^{\frac{1}{2}} - \left[\frac{e^{-2\pi inx}}{-2\pi in}\right]_{-\frac{1}{2}}^{0} \qquad \text{for } n \neq 0. \\
&= \frac{e^{-piin} - 1}{-2\pi in} + \frac{1 - e^{\pi in}}{2\pi in} \\
&= \frac{2 - (e^{-\pi in} + e^{\pi in})}{2\pi in} \\
&= \frac{2 - \Big(\cos(\pi n) - i\sin(\pi n) + \cos(\pi n) + i\sin(\pi n)\Big)}{2\pi in} \\
&= \frac{1 - \cos(\pi n)}{\pi in}. \\
c_0 &= \int_0^{\frac{1}{2}} 1\,dx \;-\; \int_{-\frac{1}{2}}^{0} 1\,dx \\
&= \Big[x\Big]_0^{\frac{1}{2}} - \Big[x\Big]_{-\frac{1}{2}}^{0} \\
&= \left(\frac{1}{2} - 0\right) - \left(0 - \left(-\frac{1}{2}\right)\right) \quad = \quad 0.
\end{aligned}$$

So the Fourier series for f is

$$\sum_{n\in\mathbb{Z},n\neq 0} \frac{1-\cos(\pi n)}{\pi in}\, e^{2\pi inx} \quad = \quad \sum_{n\in\mathbb{Z},n\neq 0} \frac{1-(-1)^n}{\pi in}\, e^{2\pi inx}$$

Notice: if n is odd, then $1 - (-1)^n = 1 - (-1) = 2$, but if n is even, then $1 - (-1)^n = 1 - (1) = 0$.
So the Fourier series for f is:

$$\sum_{n\in\mathbb{Z}} \frac{2}{\pi i(2n-1)}\, e^{2\pi i(2n-1)x}.$$

Now, by Parseval's Formula, $\|f\|_{L^2} = \|\hat{f}\|_{l^2}$ and so $\|f\|_{L^2}^2 = \|\hat{f}\|_{l^2}^2$

Well,

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int_{\mathbb{R}} \left|f(x)\right|^2 dx \\
&= \int_0^{\frac{1}{2}} |1|^2\, dx + \int_{-\frac{1}{2}}^0 |-1|^2\, dx \\
&= \Big[x\Big]_0^{\frac{1}{2}} + \Big[x\Big]_{-\frac{1}{2}}^0 \\
&= \left(\frac{1}{2} - 0\right) + \left(0 - \left(-\frac{1}{2}\right)\right) \quad = \quad 1
\end{aligned}$$

and

$$\begin{aligned}
\|\hat{f}\|_{l^2}^2 &= \sum_{n\in\mathbb{Z}} \left| \frac{2}{\pi i(2n-1)}\, e^{2\pi i(2n-1)x} \right|^2 \qquad \text{from part (a).} \\
&= \sum_{n\in\mathbb{Z}} \frac{4}{\pi^2(2n-1)^2} \\
&= \frac{4}{\pi^2} \sum_{n\in\mathbb{Z}} \frac{1}{(2n-1)^2} \\
&= \frac{4}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \; + \; \sum_{n=0}^{\infty} \frac{1}{\big(2(-n)-1\big)^2} \right) \\
&= \frac{4}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \; + \; \sum_{n=0}^{\infty} \frac{1}{(-1)^2(2n+1)^2} \right) \\
&= \frac{4}{\pi^2}\left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \; + \; \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \right) \\
&= \frac{4}{\pi^2}\left(2\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \right).
\end{aligned}$$

Thus,

$$\|\hat{f}\|_{l^2}^2 \quad = \quad \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \quad = \quad 1 \quad = \quad \|f\|_{L^2}^2$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \; = \; \frac{\pi^2}{8}.$$

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7. Assume that E is a compact Lebesgue measurable subset of $\mathbb{R}$, and let

$$f(t) \;=\; \int_E \cos(tx)\,dx,\; t \in \mathbb{R}.$$

Prove or disprove:
a. f has compact support.
b. f is infinitely differentiable.

a. False. Counterexample:
Let $E = [0, 2\pi]$, then E is a compact and Lebesgue measurable subset of $\mathbb{R}$. Then

$$f(t) \;=\; \int_0^{2\pi} \cos(tx)\,dx \;=\; \frac{1}{t}\sin(tx)\Big|_{x=0}^{2\pi} \;=\; \frac{1}{t}\sin(2\pi t) - 0, \quad \text{provided } t \neq 0.$$

If $t = 0$ then $f(0) \;=\; \int_0^{2\pi} 1\,dx \;=\; 2\pi.$

Thus, $f(t) = 0$ when $\frac{1}{t}\sin(2\pi t) = 0$, so $\sin(2\pi t) = 0$, which happens only when $t \in \mathbb{Z}$. Now, the support of f is the closure of the set of all $t \in \mathbb{R}$ such that $f(t) \neq 0$, is the closure of the complement of $\mathbb{Z}$ in $\mathbb{R}$, which is all of $\mathbb{R}$, which is NOT compact.

b. True.
One method (probably not the best method): Proof by induction.

$$f^{(n)}(t) = \begin{cases} \displaystyle\int_E (-1)^{\frac{n}{2}} x^n \cos(tx)\,dx & \text{if } n \text{ is even} \\ \displaystyle\int_E (-1)^{\frac{n+1}{2}} x^n \sin(tx)\,dx & \text{if } n \text{ is odd} \end{cases}$$

Using the Mean Value Theorem and the Lebesgue Dominated Convergence Thoerem (as in April 2013 #6b and September 2011 #7b), first show true for base cases: $n = 1$ and $n = 1$, then assuming true for the n^{th} case, show true for the $(n+1)^{\text{st}}$ and $(n+2)^{\text{nd}}$ cases.

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