Part A

1. Assume that $f \in L^1(\mathbb{R})$. Show that $\int_{-y}^{y} f(x) \, dx \to \int_{-\infty}^{\infty} f(x) \, dx$ as $y \to \infty$.

Let $\{x_n\}$ be any sequence of positive numbers such that $x_n \to \infty$ as $n \to \infty$. Define $g_n(x) := \chi_{(-x_n,x_n)} f(x)$.

Claim: $g_n \to f$ pointwise.

pf: Let $p_0 \in \mathbb{R}$. Since $x_n \to \infty$, there exists some $M$ such that $x_n = |x_n| > |p_0|$, for every $n \geq M$. So, $-x_n < p_0 < x_n$, for every $n \geq M$. Thus, $g_n(p_0) = f(p_0)$, for every $n \geq M$.

Note also that $|g_n(x)| \leq |f(x)|$, for every $x$ and for every $n$.

Now, since $|f|$ is non-negative and $f \in L^1(\mathbb{R})$, we have that $\int_{\mathbb{R}} |f(x)| \, dx < \infty$. So, by the Lebesgue Dominated Convergence Theorem we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} g_n(x) \, dx = \int_{\mathbb{R}} \lim_{n \to \infty} g_n(x) \, dx = \int_{\mathbb{R}} f(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx.$$ 

Also, by the definition of $\{g_n(x)\}$, we have

$$\lim_{n \to \infty} \int_{\mathbb{R}} g_n(x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} \chi_{(-x_n,x_n)} f(x) \, dx = \lim_{n \to \infty} \int_{-x_n}^{x_n} f(x) \, dx = \lim_{y \to \infty} \int_{-y}^{y} f(x) \, dx.$$ 

Thus,

$$\lim_{y \to \infty} \int_{-y}^{y} f(x) \, dx = \int_{-\infty}^{\infty} f(x) \, dx.$$ 

■
Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, and assume that $f' \in L^4([0, 1])$ and that $f(0) = 0$. Find all values of $\alpha$ so that

$$\lim_{x \to 0^+} x^{-\alpha} f(x) = 0$$

for all such $f$.

Since $f$ is absolutely continuous on $\mathbb{R}$, it is also absolutely continuous on $[0, x]$, for any $x \in \mathbb{R}$, but let’s only consider $x \in [0, 1]$. Also, $f'$ exists almost everywhere and is integrable on $[0, x]$, and, by the Fundamental Theorem of Calculus, $\int_0^x f'(t) \, dt = f(x) - f(0) = f(x)$. Also, note that $f' \in L^4([0, 1])$ implies that $(\int_0^1 |f'(t)|^4 \, dt)^{1/4} < \infty$.

Now,

$$0 \leq \left| \lim_{x \to 0^+} x^{-\alpha} f(x) \right| = \lim_{x \to 0^+} \left| x^{-\alpha} \right| \cdot |f(x)|$$

$$= \lim_{x \to 0^+} x^{-\alpha} \left| \int_0^x f'(t) \, dt \right|$$

$$\leq \lim_{x \to 0^+} x^{-\alpha} \left| 1 \cdot f'(t) \right| \, dt$$

$$= \lim_{x \to 0^+} x^{-\alpha} \left\| f'(t) \right\|_1 \text{ on } [0, x]$$

$$\leq \lim_{x \to 0^+} x^{-\alpha} \left\| 1 \right\|_{4/3} \cdot \left\| f'(t) \right\|_4 \quad \text{By Hölder’s Inequality}$$

$$= \lim_{x \to 0^+} x^{-\alpha} \left( \int_0^x |1|^{4/3} \, dt \right)^{3/4} \left( \int_0^x |f'(t)|^4 \, dt \right)^{1/4}$$

$$= \lim_{x \to 0^+} x^{-\alpha} x^{3/4} \left( \int_0^x |f'(t)|^4 \, dt \right)^{1/4}$$

$$= \lim_{x \to 0^+} x^{3/4-\alpha} \left( \int_{\mathbb{R}} \chi_{[0,x]} |f'(t)|^4 \, dt \right)^{1/4}$$

Now, let $\{x_n\}$ be a sequence of positive real numbers in $[0, 1]$ such that $x_n \to 0$ as $n \to \infty$. Define $g_n(x) := \chi_{[0,x_n]} |f'(x)|^4$. Then, $g_n(x) \to 0$ as $n \to \infty$, and $g_n(x) \leq |f'(x)|^4$, which is integrable, so by the Dominated Convergence Theorem, we have the following:

$$\lim_{x \to 0^+} \int_{\mathbb{R}} \chi_{[0,x]} |f'(t)|^4 \, dt = \lim_{n \to \infty} \int_{\mathbb{R}} g_n(t) \, dt = \int_{\mathbb{R}} \lim_{n \to \infty} g_n(t) \, dt = \int_{\mathbb{R}} 0 \, dt = 0.$$

Thus, $\lim_{x \to 0^+} x^{-\alpha} f(x) \leq \lim_{n \to \infty} x_n^{3/4-\alpha} \left( \int_{\mathbb{R}} g_n(t) \, dt \right)^{1/4} = 0$, provided $x_n^{3/4-\alpha} \to \infty$. So, since $x_n \to 0$, it must be that $\frac{3}{4} - \alpha \geq 0$.

Therefore, if $\alpha \leq \frac{3}{4}$, then $\lim_{x \to 0^+} x^{-\alpha} f(x) = 0$, as required. \qed
3. Recall that, if \((X, \rho)\) and \((Y, \sigma)\) are metric spaces, a function \(f : X \to Y\) is called Lipschitz continuous if there exists a constant \(\lambda \geq 0\) such that
\[
\sigma(f(x), f(y)) \leq \lambda \rho(x, y) \quad \text{for all} \quad x, y \in X.
\]
Let \(X\) be a compact metric space, and let \(Y = \mathbb{R}\). Show that the Lipschitz continuous functions are dense in the space of continuous functions with the uniform norm.

Stone-Weierstrass Theorem:
Suppose \(X\) is a compact Hausdorff space and \(A\) is a subalgebra of \(C(X, \mathbb{R})\), which contains a non-zero constant function. Then \(A\) is dense in \(C(X, \mathbb{R})\) if, and only if, it separates points.

In our case: \(A = Lip(X, \mathbb{R}) = \{f : X \to \mathbb{R} | f \text{ is Lipschitz continuous}\}\).

We will need to show the following:
1. \(Lip(X, \mathbb{R})\) is a subalgebra of \(C(X, \mathbb{R})\):
   a. \(Lip(X, \mathbb{R}) \subseteq C(X, \mathbb{R})\).
   b. \(Lip(X, \mathbb{R})\) is a linear space.
   c. \(Lip(X, \mathbb{R})\) is closed under multiplication.
2. There is a non-zero constant function in \(Lip(X, \mathbb{R})\).
3. \(Lip(X, \mathbb{R})\) separates points.
4. \(X\) is Hausdorff.

Now, let’s get to work...

1. (a) Let \(x, y \in X\) and \(f \in Lip(X, \mathbb{R})\), then \(|f(x) - f(y)| \leq \lambda \rho(x, y)\).
   Now, given \(\varepsilon > 0\), there exists \(\delta = \frac{\varepsilon}{\lambda}\), such that \(\rho(x, x_0) < \delta\).
   Then \(|f(x) - f(x_0)| \leq \lambda \rho(x, x_0) < \lambda \delta = \varepsilon\). Thus, \(Lip(X, \mathbb{R}) \subseteq C(X, \mathbb{R})\).
   (b) Let \(a, b \in \mathbb{R}\), \(f, g \in Lip(X, \mathbb{R})\), \(x, y \in X\), then
   \[
   \left| (af + bg)(x) - (af + bg)(y) \right| = \left| a[f(x) - f(y)] + b[g(x) - g(y)] \right|
   \leq |a||f(x) - f(y)| + |b||g(x) - g(y)|
   \leq |a|\lambda_1 \rho(x, y) + |b|\lambda_2 \rho(x, y)
   = \left( |a|\lambda_1 + |b|\lambda_2 \right) \rho(x, y),
   \]
   where \(|a|\lambda_1 + |b|\lambda_2 \geq 0\).
   Thus, \((af + bg) \in Lip(X, \mathbb{R})\), so \(Lip(X, \mathbb{R})\) is a linear space.
(c) Let \( x, y \in X, f, g \in \text{Lip}(X, \mathbb{R}) \). Since \( X \) is compact and \( f, g \in C(X, \mathbb{R}) \), there is some \( M \in \mathbb{R} \) such that \(|f(t)| \leq M \) and \(|g(t)| \leq M\), for every \( t \in X \). Then,
\[
|fg(x) - fg(y)| = |f(x)g(x) - f(y)g(y)| = |f(x)g(x) - [f(y)g(x) - f(y)g(x)] - f(y)g(y)|
\]
\[
= |g(x)[f(x) - f(y)] - f(y)[g(x) - g(y)]| 
\]
\[
\leq M|f(x) - f(y)| + M|g(x) - g(y)| 
\]
\[
\leq M \lambda_1 \rho(x, y) + M \lambda_2 \rho(x, y) 
\]
\[
= M(\lambda_1 + \lambda_2) \rho(x, y), \text{ where } M(\lambda_1 + \lambda_2) \geq 0. 
\]

Thus, \( fg \in \text{Lip}(X, \mathbb{R}) \), so \( \text{Lip}(X, \mathbb{R}) \) is closed under multiplication.

2. Let \( x, y \in X \), and let \( f = c \neq 0, c \in \mathbb{R} \).

Consider \(|f(x) - f(y)| = |c - c| = 0 = \lambda \rho(x, y)\), for \( \lambda = 0 \leq 0 \). Thus, \( f \in \text{Lip}(X, \mathbb{R}) \).

3. Let \( x, y \in X \) such that \( x \neq y \), and suppose \( f(t) = \rho(t, q) \), for every \( t \in X \), where \( q \in X \). Then \(|f(x) - f(y)| = |\rho(x, q) - \rho(y, q)| = |\rho(x, q) - \rho(q, y)| \leq \rho(x, y)| \). Thus, \( f \in \text{Lip}(X, \mathbb{R}) \).

Also, for \( r \in X \) such that \( r \neq q \), we have that \( f(r) = \rho(r, q) > 0 \), whereas \( f(q) = \rho(q, q) = 0 \). Thus, \( f \) separates points.

4. A topological space \( T \) is Hausdorff if for \( x, y \in T \) such that \( x \neq y \), there are disjoint open sets \( U, V \subseteq T \) with \( x \in U \) and \( y \in V \).

Now, given that \( X \) is compact and \( \mathbb{R} \) is Hausdorff, let \( f \in \text{Lip}(X, \mathbb{R}) \) and \( f(x) \neq f(y) \) in \( \mathbb{R} \). Then, \( f(x) \in U \) and \( f(y) \in V \) for open \( U, V \subseteq \mathbb{R} \) such that \( U \cap V = \emptyset \).

Since \( \text{Lip}(X, \mathbb{R}) \subseteq C(X, \mathbb{R}) \) we know that \( f^{-1}(U) \) and \( f^{-1}(V) \) are both open in \( X \).

If \( t \in f^{-1}(U) \cap f^{-1}(V) \), then \( t \in f^{-1}(U) \) and \( t \in f^{-1}(V) \), which implies that \( f(t) \in U \) and \( f(t) \in V \), but \( U \cap V = \emptyset \). Thus, \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \).

Therefore, \( X \) is Hausdorff.

Thus, by the Stone-Weierstrass Theorem, the Lipschitz continuous functions are dense in the space of continuous functions with the uniform norm, as required.\[\blacksquare\]
4. Using an appropriate Fourier series, show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

Let \( f(x) = x \) on \( (-\frac{1}{2}, \frac{1}{2}) \), which has period one, be defined on \( \mathbb{R} \). The Fourier series for \( f \) is

\[
\sum_{n \in \mathbb{Z}} c_n e^{2\pi inx}, \quad \text{where} \quad c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) e^{-2\pi inx} \, dx.
\]

So,

\[
c_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x \, dx = 0
\]

and for \( n \neq 0 \),

\[
c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} xe^{-2\pi inx} \, dx
\]

\[
= \left[ \frac{-xe^{-2\pi inx}}{2\pi in} \right]_{x=-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi inx} \, dx
\]

\[
= \left[ \frac{-xe^{-2\pi inx}}{2\pi in} - \frac{e^{-2\pi inx}}{(2\pi in)^2} \right]_{x=-\frac{1}{2}}^{\frac{1}{2}}
\]

\[
= \frac{-e^{-\pi in}}{4\pi in} - \frac{e^{\pi in}}{4\pi in} - \frac{e^{-\pi in}}{(2\pi in)^2} + \frac{e^{\pi in}}{(2\pi in)^2}
\]

\[
= 2 \cos (\pi n) + \frac{2i \sin (\pi n)}{(2\pi in)^2}
\]

\[
= \frac{(-1)^n}{2 \pi in} + 0, \quad \text{since} \ n \in \mathbb{Z}.
\]

So, the Fourier series for \( f \) is

\[
\sum_{n \in \mathbb{Z}, n \neq 0} \frac{(-1)^n}{2\pi in} e^{2\pi inx}.
\]

Now, Parseval’s Identity tells us that \( \|f\|_{L^2} = \|\hat{f}\|_{L^2} \). Thus, \( \|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 \). Well,

\[
\|f\|_{L^2}^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \, dx = \frac{1}{12}
\]

and

\[
\|\hat{f}\|_{L^2}^2 = \sum_{n \in \mathbb{Z}, n \neq 0} \left| \frac{(-1)^n}{-2\pi in} e^{2\pi inx} \right|^2
\]

\[
= \sum_{n \in \mathbb{Z}, n \neq 0} \frac{1}{4\pi^2 n^2}
\]

\[
= 2 \sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2} = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

Thus,

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12} = \frac{\pi^2}{6}.
\]

\[\blacksquare\]
5. Let \( f \in L^1(\mathbb{R}) \), let \( f > 0 \), and let \( \hat{f} \) be the Fourier transform of \( f \). Compute \( \sup_{x \in \mathbb{R}} \left| \hat{f}(x) \right| \), and show that the supremum is achieved at exactly one point.

\[
\left| \hat{f}(x) \right| = \left| \int_{\mathbb{R}} f(t) e^{-2\pi i xt} \, dt \right| \leq \int_{\mathbb{R}} |f(t)| e^{-2\pi i xt} \, dt = \int_{\mathbb{R}} |f(t)| \, dt = \|f\|_1
\]

Thus, \( \sup_{x \in \mathbb{R}} \left| \hat{f}(x) \right| = \|f\|_1 \)

This supremum is achieved at \( x = 0 \). Proof:

\[
\left| \hat{f}(0) \right| = \left| \int_{\mathbb{R}} f(t) e^{-2\pi i (0) t} \, dt \right| = \left| \int_{\mathbb{R}} f(t) \, dt \right| = \|f\|_1 \quad \text{since} \quad f > 0.
\]

So, we’ve just shown that \( \sup_{x \in \mathbb{R}} \left| \hat{f}(x) \right| = \|f\|_1 = \left| \hat{f}(0) \right| = \hat{f}(0), \quad \text{since} \quad f > 0. \)

Now, we need to show that \( x = 0 \) is the only place where this supremum is achieved. Well, if \( \left| \hat{f}(x) \right| = \|f\|_1 \), then

\[
\left| \hat{f}(x) \right|^2 = \|f\|^2_1
\]

\[
\left( \int_{\mathbb{R}} f(t) e^{-2\pi i xt} \, dt \right) \left( \int_{\mathbb{R}} f(s) e^{-2\pi i xs} \, ds \right) = \left( \int_{\mathbb{R}} f(t) \, dt \right) \left( \int_{\mathbb{R}} f(s) \, ds \right)
\]

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f(t) f(s) e^{-2\pi i x(t+s)} \, dt \, ds = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) f(s) \, dt \, ds \quad \text{by Fubini’s Theorem}
\]

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f(t) f(s) \left[ e^{-2\pi i x(t+s)} - 1 \right] \, dt \, ds = 0
\]

Now, taking the real part:

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f(t) f(s) \left[ \cos (2\pi x(t+s)) - 1 \right] \, dt \, ds = 0.
\]

Since \( f(t), f(s) > 0 \) and \( \cos (2\pi x(t+s)) - 1 \leq 0 \), if the integral is equal to zero, then it must be that \( \cos (2\pi x(t+s)) - 1 = 0 \), so \( \cos (2\pi x(t+s)) = 1 \), for all \( t \) and \( s \) values. This will happen only when \( x = 0 \).

Thus, \( \left| \hat{f}(x) \right| = \hat{f}(0) = \|f\|_1 \) if, and only if, \( x = 0 \).
6. Prove or give a counterexample:

a. If \( f \in L^p(\mathbb{R}) \), for all \( 1 \leq p < \infty \), then \( f \in L^\infty(\mathbb{R}) \).

b. If \( f \) and \( g \) are Lebesgue measurable on \( \mathbb{R} \), then \( \|fg\|_4 \leq \|f\|_4 \|g\|_\infty \).

a. False. Counterexample:
Let \( f(x) = \chi_{[0,1]} \ln x \). Then \( f \in L^p(\mathbb{R}) \) for \( 1 \leq p < \infty \), but \( f \not\in L^\infty(\mathbb{R}) \).

Proof:
\[
\int_{\mathbb{R}} |f(x)|^p \, dx = \int_0^1 |\ln x|^p \, dx = \int_1^\infty \frac{|\ln y|^p}{y^2} \, dy = \int_1^\infty \frac{|\ln y|^p}{y^2} \, dy = (*)
\]

Now, \( \lim_{y \to \infty} \frac{|\ln y|^p}{\sqrt{y}} = \frac{\infty}{\infty} \), so L'Hospital's Rule applies repeatedly to get
\[
\lim_{y \to \infty} \frac{|\ln y|^p}{\sqrt{y}} = \lim_{y \to \infty} \frac{p!}{\left(\frac{1}{2}\right)^p \sqrt{y}} = 0.
\]

Thus, \( |\ln y|^p \leq M\sqrt{y} \) for some \( M \in \mathbb{R} \) and for \( y \in [1, \infty) \). Therefore,
\[
(*) \leq M \int_1^\infty \frac{\sqrt{y}}{y^2} \, dy = M \int_1^\infty y^{-\frac{3}{2}} \, dy < \infty.
\]

Thus, \( f \in L^p(I) \) for \( 1 \leq p < \infty \).

However, \( \|f\|_\infty = \|\ln x\|_\infty = \text{ess sup}_{x \in [0,1]} |\ln x| = \infty \), so \( f \not\in L^\infty(\mathbb{R}) \).

b. True. Proof:
If either \( \|f\|_4 \) or \( \|g\|_\infty \) is infinite, then there's no work to be done. So, let's assume that both \( \|f\|_4 \) and \( \|g\|_\infty \) are finite. Also, note that \( |g(x)| \leq \|g\|_\infty \) by definition.

So
\[
\|fg\|_4 = \left( \int |fg(x)|^4 \, dx \right)^{\frac{1}{4}}
\]
\[
= \left( \int |f(x)|^4 |g(x)|^4 \, dx \right)^{\frac{1}{4}}
\]
\[
\leq \left( \int |f(x)|^4 \|g\|_\infty^4 \, dx \right)^{\frac{1}{4}}
\]
\[
= \|g\|_\infty \left( \int |f(x)|^4 \, dx \right)^{\frac{1}{4}}
\]
\[
= \|g\|_\infty \|f\|_4
\]

which completes the proof. ■
7. Prove or give a counterexample:

a. Every locally compact metric space is complete.
b. Every complete metric space is locally compact.

a. False. Counterexample:
Let \( E = (0, 1) \subseteq \mathbb{R} \). Then \( E \) is a locally compact metric space since for every \( x \in (0, 1) \) there exists some interval \([a, b] \subseteq (0, 1)\) such that \( x \in [a, b] \). Let \( [a, b] = \left[ \frac{x}{2}, \frac{x + 1}{2} \right] \).
However, \( E \) is not complete: consider the sequence \[ \left\{ \frac{1}{n} \right\}_{n=2}^\infty \]
which is Cauchy since \( \left| \frac{1}{m} - \frac{1}{n} \right| \to 0 \) as \( m, n \to \infty \). However, \( \lim_{n \to \infty} \frac{1}{n} = 0 \notin E \). Thus, \( E \) is not a complete metric space.

b. False. Counterexample:
Let \( X = L^2(\mathbb{R}) \), which is a Hilbert space with infinite dimension and inner product \( \langle f, g \rangle = \int_{\mathbb{R}} fg \, d\mu \), where \( \mu \) is the Lebesgue measure. Notice: \( X \) is a complete metric space with respect to the metric induced by the norm \( \| f \| = \sqrt{\langle f, f \rangle} = \left( \int_{\mathbb{R}} f^2 \, d\mu \right)^{\frac{1}{2}} \).
Now, since \( X \) has infinite dimension, it has an infinite orthonormal basis. Let \( \{ f_k \}_{k=1}^\infty \) be a countable orthonormal set in \( X \), then \[ \langle f_j, f_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases} \]
Now, \( X \) is not locally compact:
By way of contradiction, suppose that there exists some compact neighborhood \( K \) of 0. Then, there exists some open \( U \) such that \( 0 \in U \subseteq K \). Thus, there exists some \( n > 0 \) such that the ball of radius \( n \) about 0 is contained in \( U \).
Define \( g_n = \frac{n}{2} f_n \) for every \( n \). So
\[ \langle g_n, g_m \rangle = \langle \frac{n}{2} f_n, \frac{m}{2} f_m \rangle \]
\[ = \frac{nm}{4} \langle f_n, f_m \rangle \]
\[ = \begin{cases} 0 & \text{if } n \neq m \\ \frac{n^2}{4} & \text{if } n = m \end{cases} \]
Moreover, \[ \|0 - g_n\| = \sqrt{\langle g_n, g_n \rangle} = \sqrt{\frac{n^2}{4}} = \frac{n}{2}. \]
Thus, \( g_n \) is in the ball with radius \( n \) centered at 0, so \( g_n \in U \subseteq K \).
Now, by the Bolzano-Weierstrass Theorem for compactness, \( \{g_n\} \) must contain a convergent subsequence in \( K \). However, for every \( \{g_{n_i}\} \subseteq \{g_n\} \), the following is true:

\[
\|g_{n_j} - g_{n_k}\| = \sqrt{\langle g_{n_j} - g_{n_k}, g_{n_j} - g_{n_k} \rangle} \\
= \left( \int (g_{n_j} - g_{n_k})^2 \right)^{\frac{1}{2}} \\
= \left( \int g_{n_j}^2 + \int g_{n_k}^2 - 2 \int g_{n_j} g_{n_k} \right)^{\frac{1}{2}} \\
= \left( \frac{n^2}{4} + \frac{n^2}{4} + 0 \right)^{\frac{1}{2}} \\
= \frac{n}{\sqrt{2}} \not\rightarrow 0
\]

So \( \{g_{n_k}\} \) is not Cauchy. But in a complete metric space, every convergent subsequence is Cauchy. Thus, \( \{g_{n_k}\} \) cannot converge. Therefore, \( K \), a neighborhood of 0, is not compact.

\[\blacksquare\]