Monads

Definition

A monad $T$ is a category $C$ is a monoid in the category of endofunctors $C^C$.

More explicitly, a monad in $C$ is a triple $\langle T, \eta, \mu \rangle$, where $T : C \to C$ is a functor and $\eta : I_C \to T, \mu : T \circ T \to T$ are natural transformations which make the following diagrams commute:

$$
\begin{align*}
T \circ T \circ T & \xrightarrow{T\mu} T \circ T \\
\mu T & \downarrow \\
T \circ T & \xrightarrow{\mu} T
\end{align*}
$$

$$
\begin{align*}
T & \xrightarrow{\eta T} T \circ T \\
& \xleftarrow{T \eta} T
\end{align*}
$$

Dually (by inverting the arrows in the given definition) one can define the notion of a comonad.
**Monads**

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A *monad* $T$ is a category $\mathcal{C}$ is a monoid in the category of endofunctors $\mathcal{C}^\mathcal{C}$.

More explicitly, a monad in $\mathcal{C}$ is a triple $\langle T, \eta, \mu \rangle$, where $T : \mathcal{C} \to \mathcal{C}$ is a functor and $\eta : I_\mathcal{C} \to T, \mu : T \circ T \to T$ are natural transformations which make the following diagrams commute:

\[
\begin{align*}
T \circ T \circ T & \xrightarrow{T\mu} T \circ T \\
\mu T & \downarrow \mu \\
T \circ T & \xrightarrow{\mu} T
\end{align*}
\]

\[
\begin{align*}
T & \xrightarrow{\eta T} T \circ T \\
T & \xleftarrow{T\eta}
\end{align*}
\]

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\[
\begin{align*}
T \circ T \circ T & \xrightarrow{T\mu} T \circ T & T & \xrightarrow{\eta T} T \circ T & T & \xleftarrow{T\eta} T \\
\mu T & \downarrow & \mu & \downarrow & \text{id} & \downarrow & \text{id} \\
T \circ T & \xrightarrow{\mu} T & T & & T & & T
\end{align*}
\]

Dually (by inverting the arrows in the given definition) one can define the notion of a *comonad*.
Example

1) **Trivial monad**
   In any category $\mathbf{C}$ the triple $\langle I_\mathbf{C}, id, id \rangle$ forms a monad.

2) **Topological closure**
   The functor $T : \textbf{Top} \rightarrow \textbf{Top}$ defined by $T(X) = \overline{X}$ together with the natural transformations

   $\eta : I \rightarrow T,$ \hspace{1cm} $\eta_X : X \hookrightarrow \overline{X}$

   $\mu : T \circ T \rightarrow T,$ \hspace{1cm} $\mu_X = id_X$

   forms a monad.
Monads

Example

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   In any category $\mathbf{C}$ the triple $\langle I_{\mathbf{C}}, id, id \rangle$ forms a monad.

2) **Topological closure**
   The functor $T : \mathbf{Top} \to \mathbf{Top}$ defined by $T(X) = \overline{X}$ together with the natural transformations

   \[ \eta : I \to T, \quad \eta_X : X \hookrightarrow \overline{X} \]

   \[ \mu : T \circ T \to T, \quad \mu_X = id_X \]

   forms a monad.
Example

3) The free category monad

Let **Graphs** be a category of oriented graphs and morphisms between them. There is a functor **Graphs → SmallCats** which sends a graph $\Gamma$ to its free category $F_\Gamma$.
Example

3) The free category monad

Let $\text{Graphs}$ be a category of oriented graphs and morphisms between them. There is a functor $\text{Graphs} \to \text{SmallCats}$ which sends a graph $\Gamma$ to its free category $F_\Gamma$

- objects in $F_\Gamma = \text{vertices of } \Gamma$;
- morphisms in $F_\Gamma = \text{edges of } \Gamma$
  - + identity morphisms
  - + arrows induced by finite paths in $\Gamma$. 
Example

3) *The free category monad*

Let \textbf{Graphs} be a category of oriented graphs and morphisms between them. There is a functor \textbf{Graphs} → \textbf{SmallCats} which sends a graph Γ to its free category \(F_Γ\) and the forgetful functor \textbf{SmallCats} → \textbf{Graphs}.

The composition \(F\) of these two functors yields a monad in \textbf{Graphs}. 
Example

3) The free category monad

Let \textbf{Graphs} be a category of oriented graphs and morphisms between them. There is a functor \textbf{Graphs} \rightarrow \textbf{SmallCats} which sends a graph \( \Gamma \) to its free category \( F_\Gamma \) and the forgetful functor \textbf{SmallCats} \rightarrow \textbf{Graphs}.

The composition \( F \) of these two functors yields a monad in \textbf{Graphs}.

On \( X \in \textbf{Graphs} \) the natural transformation \( \eta : I \rightarrow F \) is the injection \( X \hookrightarrow F(X) \) and \( \mu_X : (F \circ F)(X) \rightarrow F(X) \) evaluates the paths compositions.
An algebra over a monad \( T = \langle T, \eta, \mu \rangle \) (or simply a \( T \)-algebra) in a category \( \mathbf{C} \) consists of an object \( X \in \mathbf{C} \) and a morphism \( h : TX \to X \) such that the following diagrams commute:
Example

1) Any object $X \in \mathcal{C}$ is an algebra over the trivial monad in $\mathcal{C}$.

2) If $T$ is a monad in $\mathcal{C}$ and $X \in \mathcal{C}$, then $TX$ is automatically a $T$-algebra.

3) Any closed topological space is an algebra over the topological closure monad.
Example

1) Any object $X \in \mathbf{C}$ is an algebra over the trivial monad in $\mathbf{C}$.

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Algebras over monads

Example

1) Any object $X \in C$ is an algebra over the trivial monad in $C$.

2) If $T$ is a monad in $C$ and $X \in C$, then $TX$ is automatically a $T$-algebra.

3) Any closed topological space is an algebra over the topological closure monad.
Example

4) Let $G$ be a group. We define the left $G$-action monad $T = \langle T, \eta, \mu \rangle$ in $\mathbf{Sets}$ as follows:

$$T(X) = G \times X$$
$$\eta_X : X \to G \times X$$
$$x \mapsto (e, x)$$
$$\mu_X : G \times (G \times X) \to G \times X$$
$$(g_1, (g_2, x)) \mapsto (g_1g_2, x).$$

Then a $T$-algebra is a set $Y$ with the map $h : G \times Y \to Y$ satisfying certain compatibility conditions. It turns out that a $T$-algebra $Y$ is simply a set with a left $G$-action.
Example

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Then a $T$-algebra is a set $Y$ with the map $h : G \times Y \to Y$ satisfying certain compatibility conditions. It turns out that a $T$-algebra $Y$ is simply a set with a left $G$-action.
Let \( \langle F, G, \eta, \epsilon \rangle \) be an adjunction \( \mathbf{C} \xleftrightarrow{F \ G} \mathbf{D} \). By definition, for the unit \( \eta : 1_{\mathbf{D}} \to GF \) and counit \( \epsilon : FG \to 1_{\mathbf{C}} \) transformations, the compositions

\[
F \xrightarrow{F \eta} FGF \xrightarrow{\epsilon F} F
\]

\[
G \xrightarrow{\eta} GFG \xrightarrow{G \epsilon} G
\]

are the identity transformations \( 1_F \) and \( 1_G \) respectively.
The composition $T = GF$ is an endofunctor in $\mathbf{D}$ and there is a natural transformation from $T \circ T$ to $T$:

$$\mu = G\epsilon F : GFGF \to GF$$

Moreover, the following diagrams commute:

Thus, $\langle T, \mu, \eta \rangle$ is a monad in $\mathbf{D}$. 
Adjoint functors produce monads

The composition \( T = GF \) is an endofunctor in \( \mathbf{D} \) and there is a natural transformation from \( T \circ T \) to \( T \):

\[
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\]

Moreover, the following diagrams commute:

\[
\begin{array}{ccc}
GFGFGF & \xrightarrow{GFG\epsilon F} & GFGF \\
\downarrow G\epsilon F & & \downarrow G\epsilon F \\
GFGF & \xrightarrow{G\epsilon F} & GF
\end{array}
\]

\[
\begin{array}{ccc}
GFGF & \xrightarrow{\eta GF} & GFGF \\
\uparrow & & \uparrow \\
GF & \xrightarrow{id} & GF
\end{array}
\]

\[
\begin{array}{ccc}
GFI_D & \xleftarrow{GF\eta} & GFGF \\
\uparrow & & \uparrow \\
GF & \xleftarrow{id} & GF
\end{array}
\]

Thus, \( \langle T, \mu, \eta \rangle \) is a monad in \( \mathbf{D} \).
Adjoint functors produce monads

The composition $T = GF$ is an endofunctor in $D$ and there is a natural transformation from $T \circ T$ to $T$:

$$\mu = G\epsilon F : GFGF \to GF$$

Moreover, the following diagrams commute:

Thus, $\langle T, \mu, \eta \rangle$ is a monad in $D$. 
Adjoint functors produce monads

Remarks

1) In a similar way, from an adjunction $\mathcal{C} \xleftrightarrow{G \, F} \mathcal{D}$ one can construct a comonad $\langle S, \epsilon, \delta \rangle$ in the category $\mathcal{C}$.

2) In fact, any (co-)monad is induced by a certain adjunction. This is a theorem due to Eilenberg and Moore [1965].
The universal property of $\Delta$

We equip the simplicial category $\Delta$ with a monoidal structure as follows. Let the monoidal product in $\Delta$ be the bifunctor $+: \Delta \times \Delta \to \Delta$, which sends a pair $([m], [n])$ to $[n + m]$ and a pair of morphisms $f : [n] \to [n']$, $g : [m] \to [m']$ to the morphism

$$(f + g)(i) = \begin{cases} f(i), & 0 \leq i \leq n - 1 \\ n' + g(i - n), & n \leq i \leq n + m - 1 \end{cases}.$$ 

The unit object in $\Delta$ is $[0]$. The object $[1] \in \Delta$ together with the (unique) morphisms $\mu : [2] \to [1]$ and $\eta : [0] \to [1]$ is a monoid $\langle [1], \mu, \eta \rangle$ in $\Delta$.

**Theorem**

Let $\langle B, \otimes, I \rangle$ be a (strictly) monoidal category and $\langle C, \mu', \eta' \rangle$ be a monoid in $B$. Then there exists a unique morphism of monoidal categories $F : \langle \Delta, +, 0 \rangle \to \langle B, \otimes, I \rangle$ which sends the monoid $\langle [1], \mu, \eta \rangle$ to $\langle C, \mu', \eta' \rangle$. 
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Theorem

Let $\langle \mathcal{B}, \otimes, I \rangle$ be a (strictly) monoidal category and $\langle C, \mu', \eta' \rangle$ be a monoid in $\mathcal{B}$. Then there exists a unique morphism of monoidal categories $F : \langle \Delta, +, 0 \rangle \to \langle \mathcal{B}, \otimes, I \rangle$ which sends the monoid $\langle [1], \mu, \eta \rangle$ to $\langle C, \mu', \eta' \rangle$. 

Simplicial objects from comonads.

Let $T$ be a monad in $\mathbf{D}$, that is, $T$ is a monoid in the category $\mathbf{D}^\mathbf{D}$. By the previous theorem, there is a unique morphism $\Delta \to \mathbf{D}^\mathbf{D}$ which sends $[1]$ to $T$. Thus, a monad $T$ gives a cosimplicial object in $\mathbf{D}^\mathbf{D}$. By duality, a comonad $\langle L, \delta, \epsilon \rangle$ in $\mathbf{D}$ yields an augmented simplicial object $\Delta^{op} \to \mathbf{D}^\mathbf{D}$, which we denote by $\text{Smp}(L)$:

$$
\begin{array}{cccccccc}
I & \xleftarrow{\epsilon} & L & \xleftarrow{d_0} & L_2 & \xleftarrow{d_0} & L_3 & \xleftarrow{d_1} & L_4 & \ldots
\end{array}
$$

where

$$
d^n_i = L^i \epsilon L^{n-i} : L^{n+1} \to L^n
$$

$$
s^n_i = L^i \delta L^{n-i-1} : L^n \to L^{n+1}
$$
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$$
\begin{align*}
1 & \xrightarrow{\epsilon} L \\
& \xleftarrow{d_0} L \xrightarrow{s_0} L^2 \\
& \xleftarrow{d_1} L^2 \xrightarrow{s_1} L^3 \\
& \xleftarrow{d_2} L^3 \xrightarrow{s_0} L^4 \xrightarrow{s_1} \ldots
\end{align*}
$$

where

$$
\begin{align*}
d^n_i & = L^i \epsilon L^{n-i} : L^{n+1} \to L^n \\
s^n_i & = L^i \delta L^{n-i-1} : L^n \to L^{n+1}
\end{align*}
$$
Simplicial objects from comonads.

Let $T$ be a monad in $\mathbf{D}$, that is, $T$ is a monoid in the category $\mathbf{D}^\mathbf{D}$. By the previous theorem, there is a unique morphism $\Delta \to \mathbf{D}^\mathbf{D}$ which sends $[1]$ to $T$. Thus, a monad $T$ gives a cosimplicial object in $\mathbf{D}^\mathbf{D}$. By duality, a comonad $\langle L, \delta, \epsilon \rangle$ in $\mathbf{D}$ yields an augmented simplicial object $\Delta^{op} \to \mathbf{D}^\mathbf{D}$, which we denote by $Smp(L)$:

$$
\begin{array}{cccccccc}
I & \xleftarrow{\epsilon} & L & \xleftarrow{s_0} & L^2 & \xleftarrow{s_0} & L^3 & \xleftarrow{d_2} & L^4 & \cdots \\
 & \xleftarrow{d_0} & & \xleftarrow{d_0} & & \xleftarrow{d_0} & & \xleftarrow{d_0} & & \\
 & \xleftarrow{d_1} & \Rightarrow & \xleftarrow{d_1} & \Rightarrow & \xleftarrow{d_1} & \Rightarrow & \xleftarrow{d_1} & \Rightarrow
\end{array}
$$

where

$$
d^m_i = L^i \epsilon L^{n-i} : L^{n+1} \to L^n \quad s^n_i = L^i \delta L^{n-i-1} : L^n \to L^{n+1}
$$
Simplicial objects from comonads.

Now, if $\mathbf{D}$ is an additive category (or there is a functor $F$ from $\mathbf{D}^D$ to an additive category), then for any $X \in \mathbf{D}$, the simplicial identities would imply that the alternating sums

$$\partial_n = d_0^n - d_1^n + \cdots + (-1)^n d_n^n : L^{n+1}(X) \to L^{n+1}(X)$$

satisfy $\partial_n \partial_{n+1} = 0$.

That gives a chain complex

$$L(X) \leftarrow \partial L^2(X) \ldots \leftarrow \partial L^n(X) \ldots$$

with the augmentation $\epsilon_X : L(X) \to X$.

This complex is called the standard or bar resolution of $X$. 
Now, if $\mathbf{D}$ is an additive category (or there is a functor $F$ from $\mathbf{D}^\mathbf{D}$ to an additive category), then for any $X \in \mathbf{D}$, the simplicial identities would imply that the alternating sums

$$\partial_n = d^n_0 - d^n_1 + \cdots + (-1)^n d^n_n : L^{n+1}(X) \to L^{n+1}(X)$$

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That gives a chain complex

$$L(X) \leftarrow \partial \quad L^2(X) \ldots \leftarrow \partial \quad L^n(X) \ldots$$

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This complex is called the standard or bar resolution of $X$. 
Simplicial objects from comonads.

Now, if $D$ is an additive category (or there is a functor $F$ from $D^D$ to an additive category), then for any $X \in D$, the simplicial identities would imply that the alternating sums

$$\partial_n = d_0^n - d_1^n + \cdots + (-1)^n d_n^n : L^{n+1}(X) \to L^{n+1}(X)$$

satisfy $\partial_n \partial_{n+1} = 0$.

That gives a chain complex

$$L(X) \leftarrow^\partial L^2(X) \cdots \leftarrow^\partial L^n(X) \cdots$$

with the augmentation $\epsilon_X : L(X) \to X$.

This complex is called the standard or bar resolution of $X$. 
Simplicial objects from comonads.

Adjoint pair $\xrightarrow{\sim} \text{(Co)monad} \xrightarrow{\sim} \text{Bar-resolution}$

**Example**

Let $G$ be a group and $G \text{−} \text{Mod}$ be the category of left $G$-modules.

*Step 1.* The forgetful functor $U : G \text{−} \text{Mod} \to \text{Ab}$ has a left adjoint:

$$G \text{−} \text{Mod} \xleftarrow{\otimes \mathbb{Z}G} U \text{−} \text{Ab}$$
Simplicial objects from comonads.

Adjoint pair \( \Rightarrow \Rightarrow \Rightarrow \) (Co)monad \( \Rightarrow \Rightarrow \Rightarrow \) Bar-resolution

**Example**

Let \( G \) be a group and \( G - \text{Mod} \) be the category of left \( G \)-modules.

*Step 1.* The forgetful functor \( U: G - \text{Mod} \to \text{Ab} \) has a left adjoint:

\[
G - \text{Mod} \xrightarrow{\otimes \mathbb{Z}G} \xleftarrow{U} \text{Ab}
\]
Example

Step 2. The composition $G \rightarrow \text{Mod} \rightarrow \text{Ab} \rightarrow G \rightarrow \text{Mod}$ induces a comonad $\langle L, \delta, \epsilon \rangle$ in $G \rightarrow \text{Mod}$, where $L = \mathbb{Z}G \otimes U(-)$ and

$$\epsilon_A : \mathbb{Z}G \otimes U(A) \rightarrow A$$

$$x \otimes a \mapsto x a$$

$$\delta_A : \mathbb{Z}G \otimes U(A) \rightarrow \mathbb{Z}G \otimes U(\mathbb{Z}G \otimes U(A))$$

$$x \otimes a \mapsto x \otimes 1 \otimes a$$
Step 3. The category $G - \textbf{Mod}$ is additive. Consider $\mathbb{Z}$ as the trivial $G$-module. Then $\mathbb{Z}G \otimes U(\mathbb{Z}) \simeq \mathbb{Z}G$ and the simplicial object $Smp(L)(\mathbb{Z})$ looks like:

$$
\begin{align*}
&\mathbb{Z}G &\mathbb{Z}G \otimes 2 &\mathbb{Z}G \otimes 3 &\mathbb{Z}G \otimes 4 &\ldots \\
&s_0 &s_0 &s_0 &\ldots \\
&d_0 &d_0 &d_0 &\ldots \\
&d_1 &d_1 &d_2 \\
&\ldots &\ldots &\ldots &\ldots
\end{align*}
$$

Using this resolution, for a $G$-module $A$, we can compute the right derived functor of $\text{Hom}_{G - \textbf{Mod}}(-, A)$. That what is called the cohomology of the group $G$. 
Example

**Step 3.** The category $G - \textbf{Mod}$ is additive. Consider $\mathbb{Z}$ as the trivial $G$-module. Then $\mathbb{Z}G \otimes U(\mathbb{Z}) \simeq \mathbb{Z}G$ and the simplicial object $Smp(L)(\mathbb{Z})$ looks like:

$$
\begin{array}{cccccccc}
ZG & \overset{s_0}{\longrightarrow} & ZG \otimes 2 & \overset{s_1}{\longrightarrow} & ZG \otimes 3 & \overset{s_2}{\longrightarrow} & ZG \otimes 4 & \cdots \\
\overset{d_0}{\longleftarrow} & & \overset{d_0}{\longrightarrow} & & \overset{d_0}{\longrightarrow} & & \overset{d_0}{\longrightarrow} & \\
\overset{d_1}{\longleftarrow} & & \overset{d_1}{\longrightarrow} & & \overset{d_1}{\longrightarrow} & & \overset{d_1}{\longrightarrow} & \\
\overset{d_2}{\longleftarrow} & & \overset{d_2}{\longrightarrow} & & \overset{d_2}{\longrightarrow} & & \overset{d_2}{\longrightarrow} & \\
\end{array}
$$

Each tensor power $\mathbb{Z}G^{\otimes n+1}$ is generated by the elements of the form

$$x \otimes x_1 \otimes \cdots \otimes x_n = x[x_1| \ldots |x_n],$$

where $x, x_1, \ldots, x_n \in G$. Using this resolution, for a $G$-module $A$, we can compute the right derived functor of $\text{Hom}_{G - \textbf{Mod}}(-, A)$. That what is called the cohomology of the group $G$. 

Group cohomology

Example

Step 3. The category $G - \text{Mod}$ is additive. Consider $\mathbb{Z}$ as the trivial $G$-module. Then $\mathbb{Z}G \otimes U(\mathbb{Z}) \simeq \mathbb{Z}G$ and the simplicial object $Smp(L)(\mathbb{Z})$ looks like:

$$
\mathbb{Z}G \xrightarrow{s_0} \mathbb{Z}G \otimes 2 \xrightarrow{s_0} \mathbb{Z}G \otimes 3 \xrightarrow{s_0} \mathbb{Z}G \otimes 4 \xrightarrow{s_0} \cdots
$$

On the generators the face maps act as

$$
d_i(x_1|\ldots|x_n) = \begin{cases} 
xx_1[x_2|\ldots|x_n], & i = 0 \\
x_1[x_2|\ldots|x_{i+1}|x_{i+2}|\ldots|x_n], & 0 < i < n \\
x[x_1|\ldots|x_{n-1}], & i = n 
\end{cases}
$$

Using this resolution, for a $G$-module $A$, we can compute the right derived functor of $\text{Hom}_{G-\text{Mod}}(-, A)$. That what is called the cohomology of the group $G$. 
Step 3. The category $G - \text{Mod}$ is additive. Consider $\mathbb{Z}$ as the trivial $G$-module. Then $\mathbb{Z}G \otimes U(\mathbb{Z}) \simeq \mathbb{Z}G$ and the simplicial object $Smp(L)(\mathbb{Z})$ looks like:

The alternating sums of $d_i$ are the boundary operators of the augmented chain complex

$$\mathbb{Z} \leftarrow \mathbb{Z}G \leftarrow^\partial \mathbb{Z}G \otimes^2 (X) \ldots \leftarrow^\partial \mathbb{Z}G \otimes^n (X) \ldots$$

This is the standard or bar resolution of the trivial $G$-module $\mathbb{Z}$. Using this resolution, for a $G$-module $A$, we can compute the right derived functor of $\text{Hom}_{G-\text{Mod}}(-, A)$. That what is called the cohomology of the group $G$. 

Example

Step 3. The category $G-\text{Mod}$ is additive. Consider $\mathbb{Z}$ as the trivial $G$-module. Then $\mathbb{Z}G \otimes U(\mathbb{Z}) \simeq \mathbb{Z}G$ and the simplicial object $Smp(L)(\mathbb{Z})$ looks like:

Using this resolution, for a $G$-module $A$, we can compute the right derived functor of $\text{Hom}_{G-\text{Mod}}(-, A)$. That what is called the cohomology of the group $G$. 