Gaussian integrals and Feynman diagrams

February 28
“A mathematician is one to whom the equality \[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \] is as obvious as that twice two makes four is to you.”

Lord W.T. Kelvin to his students
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A physicist must be one to whom the formula

\[ = \bar{v}^{s'}(p')(-ie\gamma^\mu)w^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k'). \]

is equally obvious.
Introduction

Generalizing \( \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi} \):

\[
\int_{-\infty}^{\infty} P(x) e^{-\frac{x^2}{2}} \, dx
\]

- \( \int_{\mathbb{R}^n} P(x) e^{-\frac{\langle x, Ax \rangle}{2}} \, dx \) Here, \( A \) is a positively-defined symmetric matrix and \( \langle -, - \rangle \) is the standard inner product on \( \mathbb{R}^n \).
- \( \int_{\mathbb{R}^n} P(x) e^{-\frac{\langle x, Ax \rangle}{2}} - \sum_{r \geq 3} \hbar^{r/2-1} \frac{1}{r!} B_r(x, ..., x) \, dx \), where \( \hbar \) is a parameter.
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Gaussian integrals and Feynman diagrams
In *classical mechanics*, the motion of a point-particle of mass \( m \) in a potential field \( V = V(x) \) is determined by the *Newton’s equation*

\[
x'' = -\nabla V(x).
\]

Alternatively, one can state the law of motion in the form of the *stationary action principle*. One starts by introducing the *Lagrangian* of the system

\[
L(x, x', t) = (\text{kinetic energy}) - (\text{potential energy})
\]

and the *action functional*

\[
S[x] = \int_a^b L(x, x', t) \, dt.
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**Stationary action principle**

The trajectory of a particle \( x = x(t) \) (\( t \in [a, b] \)) has to be an extremum of the action functional \( S \).
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This means that such a trajectory has to be a solution of the \textit{variational} problem $\delta S = 0$. The latter can be reduced to solving the \textit{Euler-Lagrange equation} (or rather a system of EL equations). In the one-dimensional case it has the form

$$
\frac{\partial L}{\partial x} - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x'} \right) = 0, \quad x(a) = x_0, x(b) = x_1.
$$

Example

The Lagrangian for a mass on a spring system is $L = \frac{m x'^2}{2} - \frac{k x^2}{2}$. The EL equation reads: $-kx - \frac{\partial}{\partial t} mx' = 0$. That is, $mx'' = -kx$ (with some boundary conditions).
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Remark

The stationary action principle is often referred to as the principle of the *least* action, but, in general, classical trajectories do not have to minimize the action.

In classical field theory one studies not the motion of point-particles, but rather motion of a "continuum of particles" (e.g. a string, a membrane, a jet of fluid etc.). Now, the "trajectory" of a system is described by a classical field $\phi = \phi(x,t)$.

The stationary action principle still holds: one starts with the Lagrangian $L = L(\phi, \ldots)$ and looks for $\phi$'s that give the extremum to the action functional $S[\phi] = \int_D L(\phi, \ldots) \, dx \, dt$.

Example

The Lagrangian for a string has the form

$$L(z, z_x, z_t) = \frac{1}{2} \left[ m \left( \frac{\partial z}{\partial t} \right)^2 - \lambda \left( \frac{\partial z}{\partial x} \right)^2 \right].$$

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Let $\mathcal{F}$ be a space of fields (for us: scalar or vector-valued functions of space-time). An observable $f$ is a function $f : \mathcal{F} \to \mathbb{R}$.

On the quantum level, the behavior of physical systems is no longer deterministic and we cannot use the SAP directly. Instead, we just hope to find the expectation values of observables: for $f : \mathcal{F} \to \mathbb{R}$,

$$
\langle f \rangle = \frac{1}{Z} \int_{\mathcal{F}} f(\phi) e^{\frac{i}{\hbar} S[\phi]} \mathcal{D}\phi,
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where $Z = \int_{\mathcal{F}} e^{\frac{i}{\hbar} S[\phi]} \mathcal{D}\phi$ is the normalizing factor (partition function).

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\[ \langle f \rangle = \frac{1}{Z} \int_{\mathcal{F}} f(\phi) e^{\frac{i}{\hbar} S[\phi]} \mathcal{D}\phi, \]

Intuitively, we are counting contributions from all possible \( \phi \)'s, but the ones which are closer to the extrema of the action functional \( S[\phi] \) yield greater contributions (if \( \phi \) deviates a lot from a classical trajectory, then oscillations of \( e^{\frac{i}{\hbar} S[-]} \) near \( \phi \) will cancel each other out).

**Stationary phase formula**

Let \( f = f(x) \) be a smooth function with a unique critical point \( c \in (a, b) \), \( f''(c) \neq 0 \) and \( g = g(x) \) be a smooth function with vanishing derivatives at \( x = a, b \). Then

\[
\int_{a}^{b} g(x) e^{\frac{i}{\hbar} f(x)} \, dx = \hbar^{1/2} e^{i f(c)/\hbar} I(\hbar),
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where \( I \) is a smooth function on \([0, \infty)\) such that \( I(0) = \sqrt{2\pi} e^{\pm \pi i/4} \frac{g(c)}{\sqrt{|f''(c)|}}. \)
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Gaussian integrals and Feynman diagrams
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The (very) **bad** thing about the formula
\[ \langle f \rangle = \frac{1}{Z} \int_{\mathcal{F}} f(\phi) e^{i\frac{i}{\hbar}S[\phi]} \mathcal{D}\phi: \]

*since the space of fields $\mathcal{F}$ is, in general, huge (infinitely-dimensional), $\mathcal{D}\phi$ is ill-defined.*

Yet, it makes sense to approximate such integrals by their finite-dimensional analogs. This leads to the problem of studying oscillating integrals of the form

\[ \int_{\mathbb{R}^n} P(x) e^{i\frac{i}{\hbar}S[x]} \, dx \]

and their real counterparts

\[ \int_{\mathbb{R}^n} P(x) e^{-\frac{1}{\hbar}S[x]} \, dx \]
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Steepest descent and stationary phase

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Let $f = f(x)$ be a smooth function with a unique critical point $c \in (a, b)$, $f''(c) \neq 0$ and $g = g(x)$ be a smooth function with vanishing derivatives at $x = a, b$. Then

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where $I$ is a smooth function on $[0, \infty)$ such that $I(0) = \sqrt{2\pi} e^{\pm \frac{\pi i}{4}} \frac{g(c)}{\sqrt{|f''(c)|}}$.

Steepest descent formula

Let $f = f(x)$ be a smooth function with a unique minimum point $c \in (a, b)$, $f''(c) > 0$ and $g = g(x)$ be a smooth function.

Then

$$\int_a^b g(x) e^{-\frac{1}{\hbar}f(x)} \, dx = \hbar^{1/2} e^{-f(c)/\hbar} I(\hbar),$$

where $I$ is a smooth function on $[0, \infty)$ such that $I(0) = \sqrt{2\pi} \frac{g(c)}{\sqrt{|f''(c)|}}$. 
Steepest descent and stationary phase

Let $B$ be a box region in $\mathbb{R}^n$ and $f = f(x)$ be a smooth function with a unique critical point $c \in B$ such that...

**Stationary phase formula**

... $|Hf(c)| \neq 0$. Then if $g = g(x)$ is a smooth function with vanishing derivatives at the boundary of $B$, we have

$$\int_B g(x)e^{\frac{i}{\hbar}f(x)} \, dx = \hbar^{n/2}e^{i\ell/\hbar}I(\hbar),$$

where $I$ is a smooth function on $[0, \infty)$ such that $I(0) = \left(\frac{2\pi}{2}\right)^{n/2}e^{\pm\pi i\sigma/4}g(c)\sqrt{|Hf(c)|}$

**Steepest descent formula**

... $Hf(c) > 0$. Then if $g = g(x)$ is a smooth function, we have

$$\int_B g(x)e^{-\frac{1}{\hbar}f(x)} \, dx = \hbar^{n/2}e^{-f(c)/\hbar}I(\hbar),$$

where $I$ is a smooth function on $[0, \infty)$ such that $I(0) = \left(\frac{2\pi}{2}\right)^{n/2}\frac{g(c)}{\sqrt{|Hf(c)|}}$
Asymptotic expansion

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\int_B g(x)e^{-\frac{1}{\hbar}f(x)} \, dx = \hbar^{n/2} e^{-f(c)/\hbar} I(\hbar) \]

**What do we want:** find a power series expansion for
\[ I(\hbar) = A_0 + A_1 \hbar + A_2 \hbar^2 + \ldots \]

*(Not so) bad thing:* Although, \( I = I(\hbar) \) is smooth, it is not analytic at in general: the Taylor series for \( I \) at \( \hbar = 0 \) may have the zero radius of convergence.

**How to fix it:** We still can try to find a *formal* power series expansion for \( I \). Then under some pretty mild conditions on \( A_i \)'s there is canonical way to ”sum“ the series up (key term: Borel summation).

Thus, we can focus on finding the power series coefficients \( A_i \).
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Wick’s theorem

**Theorem [Gian-Carlo Wick]**

Let $A$ be a symmetric, positively-defined $d$-by-$d$ matrix and $l_i$’s be linear forms on $\mathbb{R}^d$ ($i = 1, 2, \ldots, m$). Then

$$
\int_{\mathbb{R}^d} l_1(x) \ldots l_m(x) e^{-\langle x, Ax \rangle / 2} \, dx = \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \sum_{\sigma} \prod_i \langle l_i, A^{-1} l_{\sigma(i)} \rangle,
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where $\sigma$’s run through all pairings on $\{1, 2, \ldots, m\}$ (that is, $\sigma$ is an involution on $\{1, 2, \ldots, m\}$).
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Example

$m = 0, \ d = 1, \ A = 1$: $\int_{\mathbb{R}} e^{-x^2/2} \, dx = \sqrt{2\pi}$
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**Example**

$m = 2, d = 2, A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, l_1 = x, l_2 = y$:

$$\int_{\mathbb{R}^2} xy \cdot e^{-(x^2 + xy + y^2)} \, dx \, dy = \frac{2\pi}{\sqrt{3}} \cdot \frac{-1}{3}.$$
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**Example**

$m = 4$:

$$
\int_{\mathbb{R}^d} (\ldots) \sim \langle l_1, A^{-1} l_2 \rangle \cdot \langle l_3, A^{-1} l_4 \rangle + \langle l_1, A^{-1} l_3 \rangle \cdot \langle l_2, A^{-1} l_4 \rangle + \langle l_1, A^{-1} l_4 \rangle \cdot \langle l_2, A^{-1} l_4 \rangle
$$
Wick’s theorem

It is convenient to represent pairings using graphs. Namely, for a pairing $\sigma$ on a set $\{1, 2, \ldots, m\}$, consider a graph with vertices indexed by $1, 2, \ldots, m$ and connect vertices $i, \sigma(i)$ with an edge.

**Example**

The graph

![Graph Image](image)

represents the pairing $\langle 1, 3 \rangle, \langle 2, 5 \rangle, \langle 4, 6 \rangle$. 
Wick’s theorem

Example

\[
\left( \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \right)^{-1} \int_{\mathbb{R}^d} l_1(x) \ldots l_4(x) e^{-\langle x, A x \rangle / 2} \, dx =
\]

\[= F\left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) + F\left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \]

\[+ F\left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \]
Wick’s theorem: sketch of the proof

The main idea: reduce everything to the one-dimensional case, where all computations can be performed in terms of the Gamma-function.

Useful facts:

- Any real symmetric matrix $A$ can be diagonalized.
- Polarization identity. Let $P : V \times \cdots \times V \to \mathbb{R}$ be a symmetric polylinear form. Then

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{I \subset \{1, \ldots, n\}} (-1)^{n-|I|} P(\sum_{i \in I} x_i, \ldots, \sum_{i \in I} x_i).$$

- $\Gamma\left(\frac{2k+1}{k}\right) = \frac{(2k)!}{4^k k!} \sqrt{\pi}$
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\[ \int_{\mathbb{R}^d} l_1(x) \ldots l_m(x) e^{-\langle x, Ax \rangle/2} \, dx = \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \sum_{\sigma} \prod_i \langle l_i, A^{-1} l_{\sigma(i)} \rangle, \]

We can assume that \( m \) is even.

**Step 1.** Both sides of the desired identity are symmetric and polylinear forms with respect to \( l_i \)'s. Thus, *it suffices to prove it for \( l_1 = \cdots = l_m \).*

**Step 2.** The desired identity is stable under a linear change of variables. Thus, *we can choose a basis in such a way that \( A \) becomes diagonal.* Moreover, by rescaling, we can make \( A = E \). Then \( \langle x, Ax \rangle = x_1^2 + \cdots + x_d^2 \)

**Step 3.** We find \( \int_{-\infty}^{\infty} x^{2k} e^{-x^2/2} \, dx = \sqrt{2\pi} \frac{(2k)!}{2^k k!} \) by substituting \( y = x^2/2 \) and reducing it to the Gamma-function.
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\]

We can assume that \( m = 2k \).

Step 4. What is \( \frac{(2k)!}{2^k k!} \)? It’s the number of pairings on a set \( \{1, 2 \ldots, 2k\} \).
Back to our main problem: find the coefficients of the asymptotic power series expansion of the integral

$$\int_B P(x) e^{-S(x)/\hbar} \, dx,$$

where $P$ is a polynomial and $S$ is a smooth function having a unique minimum critical point $c \in B$. 
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Feynman’s theorem

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WLOG, we can assume \( c = 0 \), \( S(c) = 0 \). Then \( S(x) \) has a (formal, at least) power series expansion

\[ S(x) = \frac{\langle x, Ax \rangle}{2} + \sum_{r \geq 3} \frac{1}{r!} B_r(x, \ldots, x) \]

where \( B_r \)’s are symmetric polylinear forms.
Feynman’s theorem

Make a change of variables $x \mapsto \frac{x}{\sqrt{\hbar}}$:

$$\int_B l_1(x) \cdots l_m(x) e^{-S(x)/\hbar} \, dx =$$

$$\hbar^{m/2} \int_{B'} l_1(x) \cdots l_m(x) e^{-\frac{\langle x, Ax \rangle}{2} - \sum_{r \geq 3} \hbar^{r/2-1} \frac{B_r(x, \ldots, x)}{r!}} \, dx =$$

$$\hbar^{m/2} \int_{\mathbb{R}^d} l_1(x) \cdots l_m(x) e^{-\frac{\langle x, Ax \rangle}{2} - \sum_{r \geq 3} \hbar^{r/2-1} \frac{B_r(x, \ldots, x)}{r!}} \, dx + o(\hbar)$$
Feynman diagrams

**Definition**

A *Feynman diagram with m external vertices* is a graph (possibly, with loops and multiple edges) with m vertices of degree 1 labeled by 1, 2, \ldots, m and finitely many unlabeled vertices of degrees $\geq 3$.

$G_{\geq 3}(m) := \{\text{isomorphism classes of Feynman diagrams with } m \text{ external vertices}\}$. Here, an isomorphism of a labeled graph is supposed to preserve the labeling.
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Example
Feynman’s theorem

\[
\langle l_1 \ldots l_m \rangle := \hbar^{m/2} \int_{\mathbb{R}^d} l_1(x) \ldots l_m(x) e^{-\frac{\langle x, Ax \rangle}{2} - \sum_{r \geq 3} \frac{\hbar^r}{r!} - 1 \frac{B_r(x, \ldots, x)}{r!}} \, dx
\]

\[
= \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \sum_{\Gamma \in G_{\geq 3}(m)} \frac{\hbar^{b(\Gamma)}}{|\text{Aut}(\Gamma)|} F_{\Gamma}(l_1, \ldots, l_m),
\]

where

- \( b(\Gamma) = |\text{Edges of } \Gamma| - |\text{internal vertices of } \Gamma| \);
- \( |\text{Aut}(\Gamma)| \) is the number of \textit{automorphisms} of a graph \( \Gamma \) which leave the external vertices fixed;
- \( F_{\Gamma}(l_1, \ldots, l_m) \) is the \textit{Feynman amplitude} of the graph \( \Gamma \) computed as follows.
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Theorem

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- \( F_\Gamma(l_1, \ldots, l_m) \) is the **Feynman amplitude of the graph** \( \Gamma \) computed as follows.
Feynman amplitude of a graph

For a connected graph $\Gamma$,

**Step 1.** put the linear form $l_i$ at the $i$-th external vertex $(i = 1, \ldots, m)$;

**Step 2.** put the polylinear form $B_r$ at each internal vertex of degree $r$ ($r = 3, 4, \ldots$);

**Step 3.** take contractions of these forms along the edges of $\Gamma$ using the pairing $\langle -, A^{-1} - \rangle$.

If $\Gamma$ has several connected components $\Gamma_i$, $F_\Gamma$ is defined to be the product of $F_{\Gamma_i}$'s.
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If $\Gamma$ has several connected components $\Gamma_i$, $F_\Gamma$ is defined to be the product of $F_{\Gamma_i}$'s.
Example

Let $S(x) = \langle x, Ax \rangle$ (free theory). Then

$$\langle l_1 \ldots l_N \rangle = \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^b(\Gamma)}{|Aut(\Gamma)|} F_\Gamma(l_1, \ldots, l_N)$$

$$= \frac{(2\pi)^{d/2}}{\sqrt{\det A}} \sum_{\text{pairings}} \frac{\hbar^{N/2}}{1} \prod_i \langle l_i, A^{-1}l_{\sigma(i)} \rangle$$

That’s basically the statement of Wick’s theorem.
Example

Let $d = 1$, $S(x) = \frac{x^2}{2} + 4x^3$. Then

$$\langle xx xx \rangle = \sqrt{2\pi} \sum_{\Gamma \in G_3(4)} \frac{\hbar^2}{1} F_{\Gamma}(x, x, x, x)$$
Feynman diagrams in physics

\[ e^+ \rightleftharpoons \Gamma/Z^0 \rightleftharpoons e^- \]

\[ e^+ \rightleftharpoons \tilde{e}^{\pm}_{R/L} \rightleftharpoons e^- \]

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