Hilbert's third problem and Dehn's invariant

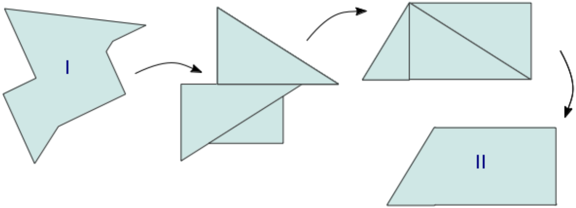
UMN Math Club

October 10

Scissors-congruence

Definition

Two polygons are said to be *scissors-congruent* if one of them can be cut into finitely many polygonal pieces that can be reassembled to give the other.



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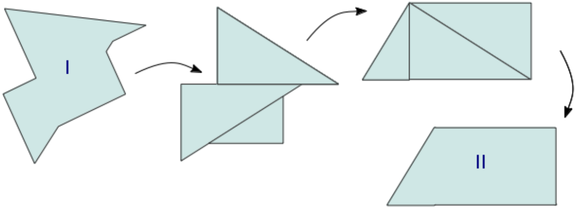

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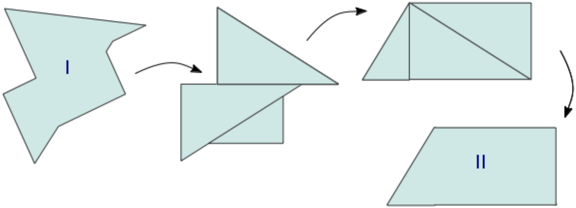

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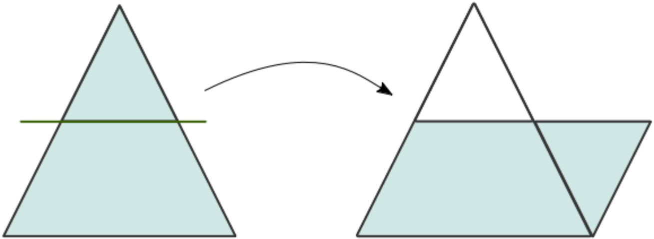
Lemma (1)

Any triangle is scissors-congruent to the square of the same area.

Proof.

Step 1. Any triangle is scissors-congruent to a parallelogram:



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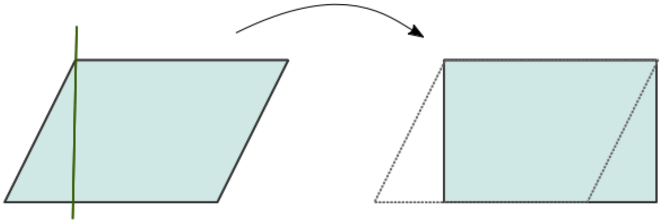

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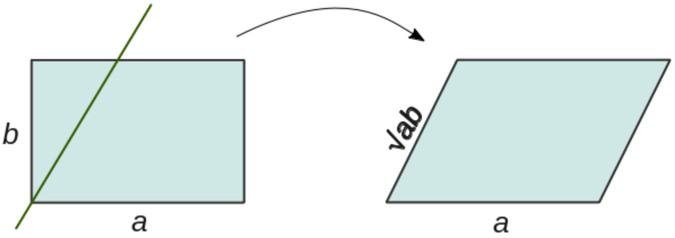

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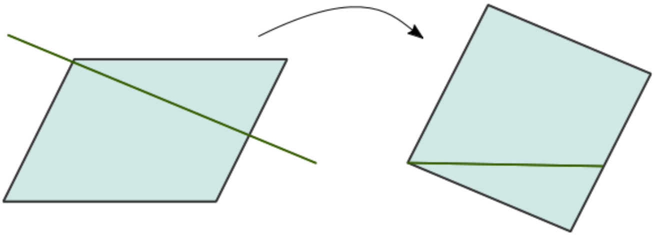
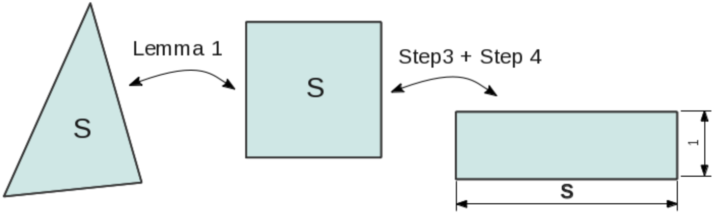


■

Scissors-congruence

Remark

By steps 3 and 4 of the lemma, a rectangle with sides of length 1 and S is scissors-congruent to the square of the same area.



Corollary

A triangle of area S is scissors-congruent to the 1-by-S rectangle.

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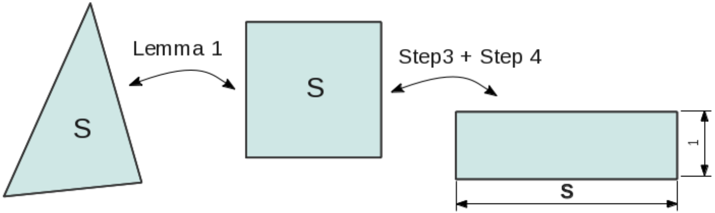

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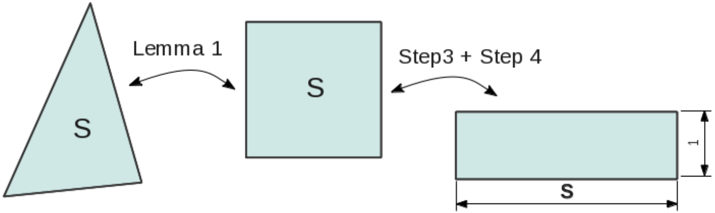

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Two polygons of the same area are scissors-congruent.

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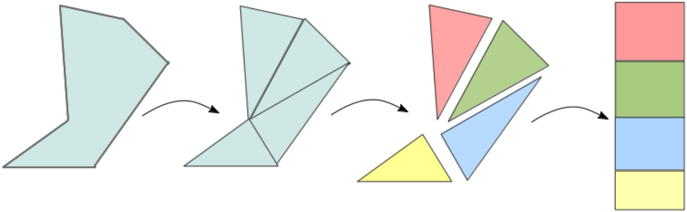

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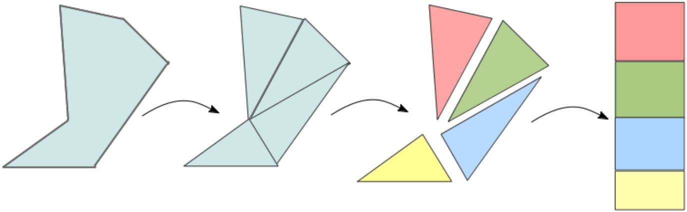

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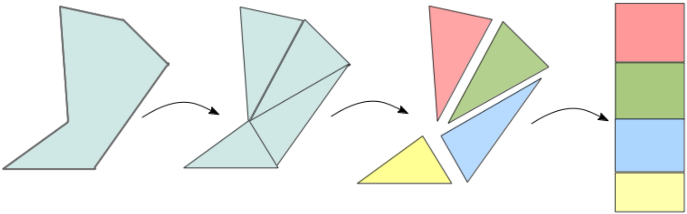

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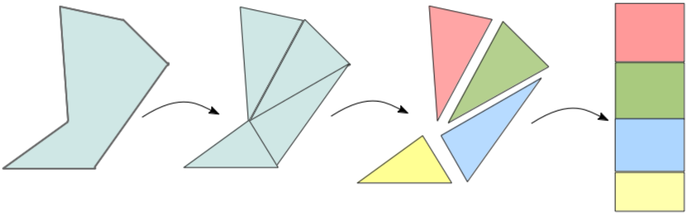

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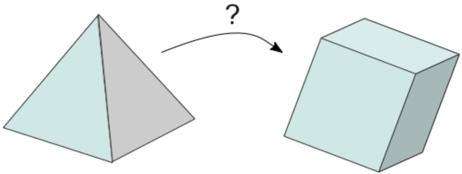

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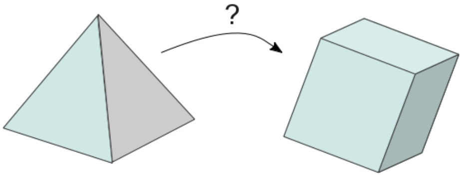

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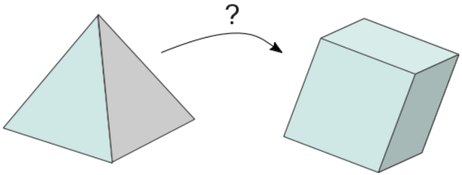

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Additive invariants of polyhedra

Theorem (M.Dehn)

A cube and a perfect tetrahedron (of the same volume) are not scissors-congruent.

The idea:

- an *additive* invariant of polyhedra is a map

$$\text{polyhedron } A \longmapsto D(A) \in \mathbb{R}$$

$$D(\underbrace{A_1 + A_2 + \cdots + A_n}_{A}) = D(A_1) + D(A_2) + \cdots + D(A_n)$$

- if polyhedra A and B are scissors-congruent, then $D(A) = D(B)$. Indeed, if A is cut into pieces $P_1, \ldots, P_n$, then

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But if we reassemble P_i's into B, we would have

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Let V be a vector space over k of possibly infinite dimension. We have the following

Theorem

Any linearly independent set $W \subset V$ is contained in some basis of V.

The case $\dim(V) < \infty$ was covered in your linear algebra class. For an infinite-dimensional V, one needs to use Zorn's lemma.

Fact

Let V and U be k-vector spaces and W be a basis of a vector space V. Defining a k-linear mapping $f : V \to U$ amounts to specifying the values of $f(\mathbf{e})$ for all basis vectors $e \in W$.

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- Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. One can always find a basis of $\mathbb{R}$ containing π.
  Pick a $\mathbb{Q}$-linear mapping $f : \mathbb{R} \to \mathbb{R}$. with the property that $f(\pi) = 0$.
- Let A be a polyhedron with the set of edges E.
  Notation:
  - $|e|$ is the length of the edge e
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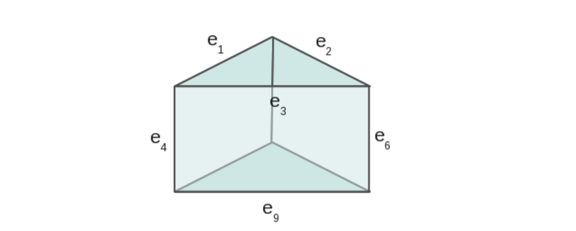

$$D(A) = |e_1| \cdot f(\alpha_{e_1}) + \cdots + |e_9| \cdot f(\alpha_{e_9})$$
$$= \sqrt{2} \cdot f(\alpha_{e_3}) + \sqrt{2} \cdot f(\alpha_{e_9}) + 1 \cdot f(\alpha_{e_1}) + \cdots + 1 \cdot f(\alpha_{e_8})$$

We have $\alpha_{e_4} = \alpha_{e_6} = \frac{\pi}{4}$ and all other angles are equal to π.
By $\mathbb{Q}$-linearity, $f\left(\frac{\pi}{4}\right) = \frac{1}{4} f(\pi) = 0$. Hence, $D(A) = 0$.

Dehn's invariant

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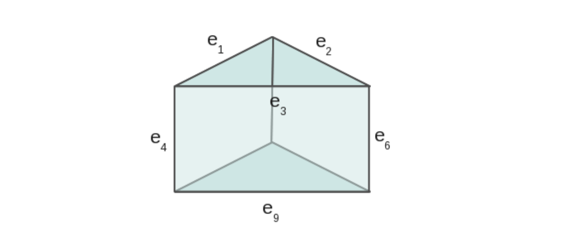

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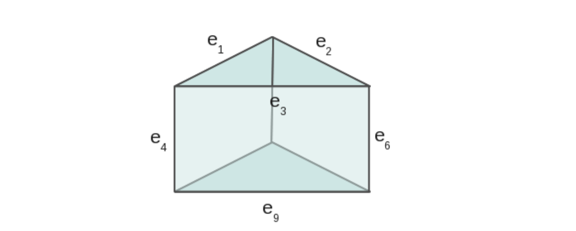

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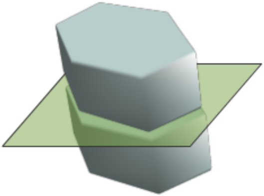
Lemma

Dehn's invariant is additive.

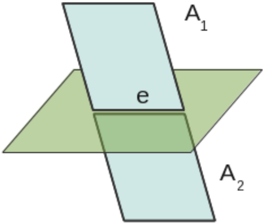
Proof. Let a polyhedron A be split into parts A_1 and A_2 by a plane L.



Let's see how the sum $D(A_1) + D(A_2)$ is related to $D(A)$.

Case 1. *Plane L intersects a face of A and creates a new edge e belonging to both A_1 and A_2.*
The contribution of e to $D(A_1) + D(A_2)$ is

$$|e| \cdot f(\alpha') + |e| \cdot f(\alpha'') = |e| \cdot (f(\alpha') + f(\alpha'')) \\ = |e| \cdot f(\alpha' + \alpha'') = |e| \cdot f(\pi) = 0.$$



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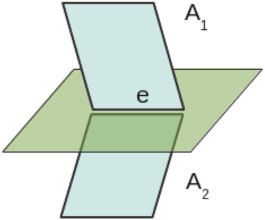
Case 2. *Plane L intersects a face of A along an edge e of A.*
The contribution of e to $D(A_1) + D(A_2)$ is

$$\begin{aligned} |e| \cdot f(\alpha') + |e| \cdot f(\alpha'') &= |e| \cdot (f(\alpha') + f(\alpha'')) \\ &= |e| \cdot f(\alpha' + \alpha'') = |e| \cdot f(\alpha_e). \end{aligned}$$

That's the same as what e contributes to $D(A)$!



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Case 3. *Plane L intersects an edge e of A splitting it into e_1 and e_2 belonging to A_1 and A_2 respectively.*
The contribution of e_1 and e_2 to $D(A_1) + D(A_2)$ is

$$|e_1| \cdot f(\alpha_e) + |e_2| \cdot f(\alpha_e) = (|e_1| + |e_2|) \cdot f(\alpha_e)$$
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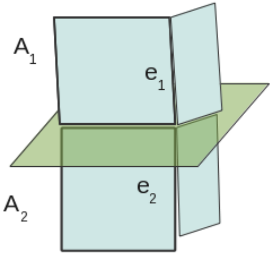

We observed that

- the new edges appearing after cutting A into two parts do not contribute anything to the sum $D(A_1) + D(A_2)$, while the contribution to this sum coming from the edges that were originally present in A is the same as to $D(A)$.

Hence, $D(A_1) + D(A_2) = D(A)$. A straightforward induction shows that additivity holds for any number of parts $A_1, \ldots, A_n$ we split A in. ■

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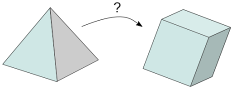

It remains to compute $D(\text{cube})$ and $D(\text{tetrahedron})$. Assume that the volume of both polyhedra is 1.

$$D(\text{cube}) = |e_1|f(\alpha_1) + \cdots + |e_{12}|f(\alpha_{12})$$
$$= 1 \cdot f(\pi/2) + \cdots + 1 \cdot f(\pi/2) = 6f(\pi) = 0$$

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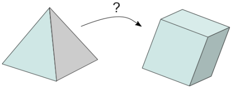


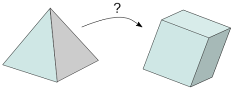
It remains to compute $D(\text{cube})$ and $D(\text{tetrahedron})$. Assume that the volume of both polyhedra is 1.

$$\begin{aligned} D(\text{cube}) &= |e_1| f(\alpha_1) + \cdots + |e_{12}| f(\alpha_{12}) \\ &= 1 \cdot f(\pi/2) + \cdots + 1 \cdot f(\pi/2) = 6 f(\pi) = 0 \end{aligned}$$

Dehn's invariant

Theorem (M.Dehn)

A cube and a perfect tetrahedron (of the same volume) are not scissors-congruent.



The dihedral angles of a perfect tetrahedron (of volume 1) are all equal to $\alpha = \arccos\left(\frac{1}{3}\right)$ and sides have length $a = \sqrt[3]{6\sqrt{2}}$.
We have

$$D(\text{tetrahedron}) = |e_1|f(\alpha_1) + \cdots + |e_6|f(\alpha_6) = 6a \cdot f(\arccos\left(\frac{1}{3}\right))$$

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Q: Can we choose f in such a way that $f(\arccos\left(\frac{1}{3}\right)) \neq 0$? In other words, is $\arccos\left(\frac{1}{3}\right)$ linearly independent (over $\mathbb{Q}$) from π?

A: Yes; $\cos\left(\frac{p}{q}\pi\right) \neq \frac{1}{3}$ (exercise).

Thus,

$$0 = D(\text{cube}) \neq D(\text{tetrahedron}) = \text{something non-zero.}$$

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Dehn's invariant

Theorem (Sydler)

Two polyhedra with the same Dehn invariants (for all f's) are scissor-congruent.

Theorem (B.Jessen, A.Throup)

The same is true for polyhedra in $\mathbb{R}^4$.