Outline

- Lattice models: definitions and examples.
- Yang-Baxter equation and integrability.
- Quantum groups and their representations.
- Integrable lattice models from intertwining operators.
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Lattice models

Consider a collection of “atoms” located at the vertices of a 2D-lattice $\mathbb{Z}_M \times \mathbb{Z}_N$:

Assumptions:

- each “atom” interacts only with its nearest neighbors;
- the energy of interaction depends only on the states of the bonds (the edges);
- bonds satisfy periodic (”toroidal“) boundary conditions.
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- the energy of interaction depends only on the states of the bonds (the edges);
- bonds satisfy periodic (”toroidal“) boundary conditions.
Denote by $\mathcal{E}_{\epsilon_1,\epsilon_2}^{\epsilon_3,\epsilon_4}$ the interaction energy of a single atom with the bonds in states $\epsilon_1, \ldots, \epsilon_4 \in \{1, \ldots, n\}$. The state of the lattice is a map 

$$\phi : \text{bonds} \rightarrow \{1, \ldots, n\}$$

The energy of such a state is $\mathcal{E}(\phi) := \sum_{\text{atoms}} \mathcal{E}_{\epsilon_1,\epsilon_2}^{\epsilon_3,\epsilon_4}$. 
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The energy of such a state is 

$$\mathcal{E}(\phi) := \sum_{\text{atoms}} \mathcal{E}_{\epsilon_1,\epsilon_2}^{\epsilon_3,\epsilon_4}$$
The **partition function** $Z = Z_{M,N}$ of such a system is

$$Z = \sum_{\text{states}} \exp(-\beta \mathcal{E}(\text{state})), \text{ where } \beta = \frac{1}{kT}.$$ 

We have

$$\exp(-\beta \mathcal{E}(\text{state})) = \exp(-\beta \sum_{\text{atoms}} \mathcal{E}_{\epsilon_3,\epsilon_4}) = \prod_{\text{atoms}} \exp(-\beta \mathcal{E}_{\epsilon_1,\epsilon_2}).$$

It what follows, it will be more convenient to work with the **Boltzmann weights**

$$R_{\epsilon_1,\epsilon_2}^{\epsilon_3,\epsilon_4} := \exp(-\beta \mathcal{E}_{\epsilon_1,\epsilon_2}^{\epsilon_3,\epsilon_4})$$

rather than with energy terms.
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It what follows, it will be more convenient to work with the Boltzmann weights

$$R^\epsilon_{\epsilon_1, \epsilon_2} := \exp(-\beta E^\epsilon_{\epsilon_1, \epsilon_2})$$

rather than with energy terms.
Standard properties of the partition function:

- \( P(\text{the system is in a state with energy } \mathcal{E}) = \frac{1}{Z} \exp(-\beta \mathcal{E}) \)
- If \( Q = Q(\phi) \) is a function of states (an "observable"), then its expectation value is

\[
\langle Q \rangle = \frac{1}{Z} \sum_{\phi \in \text{states}} Q(\phi) \exp(-\beta \mathcal{E}(\phi)).
\]
Standard properties of the partition function:

- $P(\text{the system is in a state with energy } E) = \frac{1}{Z} \exp(-\beta E)$
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Example

- $\langle \mathcal{E} \rangle = \frac{1}{Z} \sum_{\text{states}} \mathcal{E}(\phi) \exp(-\beta \mathcal{E}(\phi)) = \cdots = kT^2 \frac{\partial}{\partial T} \ln Z$
- The correlation function of edges $j_1, \ldots, j_k$ is:

$$\langle \epsilon_{j_1} \cdots \epsilon_{j_k} \rangle = \frac{1}{Z} \sum_{\text{states}} \epsilon_{j_1} \cdots \epsilon_{j_k} \exp(-\beta \mathcal{E}).$$
Partition function of a lattice model

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   \]
Examples of lattice models

1) *Eight-vertex model.*

Characteristics:

- two bond states (we denote them by + and −).
- only 8 out of 16 possible vertex configurations are allowed:

Here, $a, b, c, d$ are Boltzmann weights of the corresponding configurations.
Examples of lattice models

2) *Six-vertex model* (two-dimensional ice).
   Characteristics:
   - two bond states (we denote them by $+$ and $-$).
   - only 6 out of 16 possible vertex configurations are allowed:

\[
\begin{align*}
+ & \quad - & = & \quad - & \quad + & = & \quad =d \\
- & \quad + & = & \quad + & \quad - & = & \quad =b \\
+ & \quad - & = & \quad - & \quad + & = & \quad =c
\end{align*}
\]
Lattice models

To *solve* a model (a lattice model, in our context) means to find an explicit formula for \( Z = Z_{M,N} \), its *thermodynamical limit*

\[
\lim_{M,N \to \infty} Z_{M,N} \text{ or } \text{thermodynamical limit per site}
\]

\[
\lim_{M,N \to \infty} (Z_{M,N})^{\frac{1}{MN}}
\]

**Example**

Eight- and six-vertex models are solvable (R. Baxter, 1971).
Transfer matrix

\[ Z = \sum_{\text{states}} \exp(-\beta \mathcal{E}(\text{state})) = \sum_{\text{states}} \prod_{\text{atoms}} R_{\epsilon_1, \epsilon_2}^{\epsilon_3, \epsilon_4} \]

The contribution of a single column of the lattice to the partition function is

\[ T_{\epsilon_1'...\epsilon_N'} = \sum_{\nu_1,...,\nu_N \in \{+, -\}} R_{\nu_1 \epsilon_1}^{\nu_2 \epsilon_1'} R_{\nu_2 \epsilon_2}^{\nu_3 \epsilon_2'} ... R_{\nu_N \epsilon_N}^{\nu_1 \epsilon_N'} \]
Transfer matrix

\[ Z = \sum_{\text{states}} \exp(-\beta \varepsilon(\text{state})) = \sum_{\text{states}} \prod_{\text{atoms}} R_{\epsilon_1,\epsilon_2}^{\epsilon_3,\epsilon_4} \]

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We let $V = \mathbb{C}\langle v_+, v_- \rangle$ and regard $T_{\epsilon_1 \ldots \epsilon_N}$'s as coefficients of a linear operator ("the transfer matrix")

$$T : V \otimes N \rightarrow V \otimes N$$

$$v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_N} \mapsto \sum_{\epsilon'_i \in \{+, -\}} T_{\epsilon_1 \ldots \epsilon_N}^{{\epsilon'_1} \ldots {\epsilon'_N}} v_{\epsilon'_1} \otimes \cdots \otimes v_{\epsilon'_N}$$
Transfer matrix

Observation

The coefficients of $T \circ T$ capture the contributions from two consecutive columns; the coefficients of $T \circ T \circ T$ do this for three columns and so on.

Due to the periodic boundary conditions, we have the following Proposition

$$Z_{M,N} = \text{tr}(T^M)$$
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Observation
The coefficients of $T \circ T$ capture the contributions from two consecutive columns; the coefficients of $T \circ T \circ T$ do this for three columns and so on. Due to the periodic boundary conditions, we have the following

Proposition
$Z_{M,N} = \text{tr}(T^M)$
Let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the eigenvalues of $T$

$$Z_{M,N} = \text{tr}(T^M) = \lambda_1^M \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^M + \ldots\right).$$

So $Z \sim \lambda_1^M$ for $M \gg 0$.

Solving the six-vertex model $\rightarrow$ Solving the eigenvalue problem for $T$

The eigenvalue problem will simplify once we find a (large) family of operators commuting with $T$.

Goal

Construct a family of operators commuting with $T$. 
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$$Z_{M,N} = \text{tr}(T^M) = \lambda_1^M \left( 1 + \left( \frac{\lambda_2}{\lambda_1} \right)^M + \ldots \right).$$

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Solving the six-vertex model $\approx$ Solving the eigenvalue problem for $T$

The eigenvalue problem will simplify once we find a (large) family of operators commuting with $T$.

**Goal**

Construct a family of operators commuting with $T$. 
Remark

- The six-vertex model is equivalent (in a rather precise sense - relating the transfer matrix to the Hamiltonian) to the Heisenberg XXZ-chain model.

- This is an example of the quantum/statistical correspondence.

\[(d + 1) - \text{dimensional classical statistical model} \leftrightarrow d - \text{dimensional quantum model}\]

- Under this correspondence, the transfer matrix $T$ is analogous to the infinitesimal time evolution operator $e^{-H\Delta\tau}$.
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\[(d + 1) - \text{dimensional classical statistical model} \iff d - \text{dimensional quantum model}\]

- Under this correspondence, the transfer matrix \(T\) is analogous to the infinitesimal time evolution operator \(e^{-H\Delta\tau}\).
Working in the same vector space $V = \mathbb{C}\langle v_+, v_- \rangle$, we define

$$R : V \otimes V \rightarrow V \otimes V$$

$$v_{\epsilon_1} \otimes v_{\epsilon_2} \mapsto \sum_{\epsilon'_1, \epsilon'_2 \in \{+, -\}} R^{\epsilon'_1 \epsilon'_2}_{\epsilon_1 \epsilon_2} v_{\epsilon'_1} \otimes v_{\epsilon'_2}$$

This operator ("$R$-matrix") captures contributions of a single vertex to the partition function. Consider an $(N + 1)$-fold tensor product $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ ($V_i = V$) and let $R_{ij}$ be the operator acting on the $V_i \otimes V_j$ component of this product as $R$ and as identity on any other $V_l$. 
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\[
R_{0N} \cdots R_{02} R_{01} : V_0 \otimes V_1 \otimes \cdots \otimes V_N \rightarrow V_0 \otimes V_1 \otimes \cdots \otimes V_N
\]
Working in the same vector space $V = \mathbb{C} \langle v_+, v_- \rangle$, we define

$$R : V \otimes V \to V \otimes V$$

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Consider an $(N + 1)$-fold tensor product $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ ($V_i = V$) and let $R_{i,j}$ be the operator acting on the $V_i \otimes V_j$ component of this product as $R$ and as identity on any other $V_l$.

$$R_{0N} \ldots R_{02} R_{01} : V_0 \otimes (V_1 \otimes \cdots \otimes V_N) \to V_0 \otimes (V_1 \otimes \cdots \otimes V_N)$$
$R$-matrix

Working in the same vector space $V = \mathbb{C}\langle v_+, v_- \rangle$, we define

\[ R : V \otimes V \to V \otimes V \]

\[ v_{\epsilon_1} \otimes v_{\epsilon_2} \mapsto \sum_{\epsilon'_1, \epsilon'_2 \in \{+, -\}} R_{\epsilon_1' \epsilon_2'}^{\epsilon_1 \epsilon_2} v_{\epsilon_1'} \otimes v_{\epsilon_2'} \]

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\[ R_{0N} \cdots R_{02} R_{01} : V_0 \otimes (V_1 \otimes \cdots \otimes V_N) \to V_0 \otimes (V_1 \otimes \cdots \otimes V_N) \]

\[ L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

"monodromy matrix"
$R$-matrix

Working in the same vector space $V = \mathbb{C}\langle v_+, v_- \rangle$, we define

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$$R_{0N} \cdots R_{02} R_{01} : V_0 \otimes (V_1 \otimes \cdots \otimes V_N) \rightarrow V_0 \otimes (V_1 \otimes \cdots \otimes V_N)$$

$$L = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}$$

"monodromy matrix"

$$\text{tr}_{V_0}(R_{0N} \cdots R_{02} R_{01}) = A + D$$
**R-matrix**

Working in the same vector space $V = \mathbb{C}\langle v_+, v_- \rangle$, we define

$$R : V \otimes V \to V \otimes V$$

$$v_{\epsilon_1} \otimes v_{\epsilon_2} \mapsto \sum_{\epsilon'_1, \epsilon'_2 \in \{+, -\}} R_{\epsilon'_1 \epsilon'_2 \epsilon_1 \epsilon_2}^\epsilon v'_{\epsilon_1} \otimes v'_{\epsilon_2}$$

Consider an $(N + 1)$-fold tensor product $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ ($V_i = V$) and let $R_{ij}$ be the operator acting on the $V_i \otimes V_j$ component of this product as $R$ and as identity on any other $V_l$.

$$R_{0N} \ldots R_{02} R_{01} : V_0 \otimes (V_1 \otimes \cdots \otimes V_N) \to V_0 \otimes (V_1 \otimes \cdots \otimes V_N)$$

**Proposition**

$$T = \text{tr}_{V_0}(R_{0N} \ldots R_{02} R_{01})$$
For the six-vertex model, the $R$-matrix (in the appropriate basis of $V \otimes V$) is

$$R = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \quad (1)$$

**Theorem**

Let $R$ and $R'$ be matrices of the form (1). Define $T = \text{tr}(R_{0N} \ldots R_{01})$ and $T' = \text{tr}(R'_{0N} \ldots R'_{01})$. If there is a matrix $R''$ of the form (1) such that

$$R''_{12} R'_{13} R_{23} = R_{23} R'_{13} R''_{12} \quad (\text{on } V \otimes V \otimes V) \quad (2)$$

then $[T, T'] = 0$. Equation (2) is known as the quantum Yang-Baxter equation.
R-matrix

Example

For the six-vertex model, the $R$-matrix (in the appropriate basis of $V \otimes V$) is

$$R = \begin{bmatrix}
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**Plan of the proof.**

- Repeatedly using QYBE, show that

  $$R''_{12}L'L_{12}^{''-1} = LL',$$

  where $L = R_{0N} \ldots R_{01}$, $L' = R'_{0N} \ldots R'_{01}$ are the monodromy operators acting on $V_0 \otimes V_0' \otimes V_1 \otimes \ldots \otimes V_N$.

- Take the trace of the above identity over $V_0 \otimes V_0'$ and use the previous proposition.
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Plan of the proof.

- Repeatedly using QYBE, show that
  \[ R''_{12} L' L R''_{12}^{-1} = LL' , \]
  where $L = R_{0N} \ldots R_{01}$, $L' = R'_{0'N} \ldots R'_{0'1}$ are the monodromy operators acting on $V_0 \otimes V'_0 \otimes V_1 \otimes \ldots \otimes V_N$.
- Take the trace of the above identity over $V_0 \otimes V'_0$ and use the previous proposition.
Theorem

Let $R$ and $R'$ be matrices of the form (1). Define $T = tr_{V_0}(R_{0N} \ldots R_{01})$ and $T' = tr_{V_0'}(R'_{0N} \ldots R'_{01})$. If there is a matrix $R''$ of the form (1) such that

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Integrability

**Definition**

A hamiltonian dynamical system is said to be *completely integrable* if it has the maximal possible number of conserved quantities in involution (by Liouville, it’s $\frac{1}{2} \dim(\text{phase space})$).

By analogy, we would call a lattice model *integrable* if it admits a “large” family of operators commuting with each other and with $T$.

Due to the previous theorem, that can be formalized as follows:

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A lattice model is *integrable* if there is a family of $R$-matrices depending on parameters $\lambda, \mu, \nu$ such that for any $\mu, \nu$, there is a $\lambda$ such that

$$R_{12}(\lambda)R_{13}(\mu)R_{23}(\nu) = R_{23}(\nu)R_{13}(\mu)R_{12}(\lambda)$$
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Quantum integrability

In general, a QYBE is a system of 64 non-linear algebraic equations with 16 variables. In case of the six-vertex model, it boils down to three equations that can be solved explicitly. A family of solutions is given by

\[ R = \begin{bmatrix}
\rho \text{sh}(\eta + u) & 0 & 0 & 0 \\
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Is there a systematic way of constructing \( R \)-matrices (=integrable lattice models) for the cases other than six- or eight-vertex models?

Answer (V. Drinfeld, M. Jimbo and others) Yes.
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Similarly, there is a family of solutions of the QYBE for the eight-vertex model given in terms of elliptic functions.

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Hopf algebras and quantum groups

Definition

A coalgebra over a commutative ring $k$ is a $k$-module $A$ equipped with a comultiplication map $\Delta : A \rightarrow A \otimes A$ and a counit $A \rightarrow k$ subject to coassociativity and counitality conditions ("diagrammatical" duals of the usual associativity and unit conditions).

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Example

1. For a group $G$, a group algebra $k[G]$ equipped with a coproduct $\Delta(g) = g \otimes g$ and an antipode $S(g) = g^{-1}$ is a Hopf algebra.

2. Let $G$ be a finite group. The algebra of $k$-functions $\mathcal{F}(G)$ on $G$ is a commutative Hopf algebra with a comultiplication is $\Delta(f)(g_1, g_2) = f(g_1 g_2)$ and an antipode $S(f)(g) = f(g^{-1})$. A variation of this construction exists for compact topological groups.

3. Let $\mathfrak{g}$ be a Lie algebra over a field $k$. The universal enveloping algebra $U(\mathfrak{g})$ acquires a structure of a Hopf algebra via $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $\epsilon(x) = 0$ for $x \in \mathfrak{g}$.
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A *representation* of a Hopf algebra $A$ is a module $V$ over the *algebra* $A$.

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Hopf algebras and quantum groups

What is the meaning of the quasitriangular condition?
It’s mainly due to the following

Theorem

Let $A$ be a Hopf algebra. Then the category of $A$-modules (=$representations of $A$) is braided (=has intertwiners) if and only if $A$ is quasitriangular.

What quantum groups have to do with QYBE?

Proposition
Let $(A, \mathcal{R})$ be a quasitriangular Hopf algebra. Then the following form of QYBE holds in $A^\otimes 3$:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

An upshot: finite-dimensional representations of a quantum group give rise to an $\mathcal{R}$-matrices.
For this reason, $\mathcal{R}$ is sometimes called a universal $\mathcal{R}$-matrix.
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Quantum groups from affine Lie algebras

A rich source of quantum groups comes from affine Lie algebras.

- Start with a simple finite-dimensional Lie algebra $\mathfrak{g}$.
- Consider a central extension $\hat{\mathfrak{g}}$ of its loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.
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- Under favorable conditions, one can obtain a family of $U_q(\hat{\mathfrak{g}})$-modules $V_\zeta$, $\zeta \in \mathbb{C}$ and the universal $R$-matrix of $U_q(\hat{\mathfrak{g}})$ would give rise to the intertwining operators

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Example

The above construction applied to $\mathfrak{sl}_2$ produces $R$-matrices of the six-vertex model.