**Topic:** limit of a sequence.

1. For each of the following sequences, determine if it converges or diverges. If a sequence converges, find its limit.

   a) \( a_n = \sqrt{n^2 + n} - n \)
   
   b) \( b_n = \frac{n \sin 2n - n^2}{n \cos 2n + n^2} \)
   
   c) \( c_n = \frac{2^n}{n^2} \)
   
   d) \( d_n = \frac{5 + 0.3^n}{3 + 0.5^n} \); how about \( d'_n = \frac{5 + 0.3^n + n}{3 + 0.5^n - n} \)?
   
   e) \( e_n = \sin \ln(n+1) - \sin \ln n \)

**Solution.**

a)

\[
\lim (\sqrt{n^2 + n} - n) = \lim \left( \frac{\sqrt{n^2 + n} - n}{\sqrt{n^2 + n} + n} \right) = \lim \left( \frac{(\sqrt{n^2 + n} - n)(\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} \right) = \lim \left( \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} \right) = \lim \left( \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \right) = \frac{1}{2}.
\]

b)

\[
\lim \frac{n \sin 2n - n^2}{n \cos 2n + n^2} = \lim \left( \frac{n \sin 2n - n^2}{n \cos 2n + n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right) = \lim \left( \frac{\sin 2n - 1}{\cos 2n + 1} \right) = \lim \frac{\sin 2n - 1}{\cos 2n + 1}.
\]

To compute the limit of \( \{ \frac{\sin 2n}{n} \} \), observe that \(-1 \leq \sin 2n \leq 1\). Hence,

\[
-\frac{1}{n} \leq \frac{\sin 2n}{n} \leq \frac{1}{n}.
\]

Since \( \lim \left( -\frac{1}{n} \right) = \lim \frac{1}{n} = 0 \), then by the squeeze theorem, \( \lim \frac{\sin 2n}{n} = 0 \). Analogously, we can show that \( \lim \frac{\cos 2n}{n} = 0 \). Returning to (1), we conclude that

\[
\lim \frac{n \sin 2n - n^2}{n \cos 2n + n^2} = -1.
\]

c) We can rewrite the terms of the given sequence as \( \left( \frac{2}{n} \right)^n \). Notice that for any \( n \geq 3 \), \( \frac{2}{n} \leq \frac{2}{3} \). Therefore, \( \left( \frac{2}{n} \right)^n \leq \left( \frac{2}{3} \right)^n \). Moreover, it is clear that \( \left( \frac{2}{n} \right)^n \) is always positive. Thus we have inequalities

\[
0 \leq \left( \frac{2}{n} \right)^n \leq \left( \frac{2}{3} \right)^n.
\]

The limit of \( \{ \left( \frac{2}{n} \right)^n \} \) is equal to 0 (that’s because \(-1 < \frac{2}{3} < 1\)). Hence, by the squeeze theorem, \( \lim \left( \frac{2}{n} \right)^n = 0 \).
d) Since 0.3 and 0.5 are less than 1 by the absolute value, then \( \lim 0.3^n = \lim 0.5^n = 0 \).
Therefore,
\[
\lim d_n = \lim \frac{5 + 0.3^n}{3 + 0.5^n} = \frac{5}{3}.
\]
The limit of sequence \( d'_n = \frac{5 + 0.3^n}{3 + 0.5^n} \) can be computed as follows:
\[
\lim \frac{5 + 0.3^n + n}{3 + 0.5^n - n} = \lim \frac{(5 + 0.3^n + n) \cdot \frac{1}{n}}{(3 + 0.5^n - n) \cdot \frac{1}{n}} = \lim \frac{\frac{5}{n} + \frac{0.3^n}{n} + 1}{\frac{3}{n} + \frac{0.5^n}{n} - 1} = 0 + 0 + 1 = 0 + 0 - 1 = -1.
\]
e) We start by recalling the formula
\[
\sin A - \sin B = 2 \sin \frac{A - B}{2} \cos \frac{A + B}{2}.
\]
In our case it gives
\[
\sin \ln(n + 1) - \sin \ln n = 2 \sin \frac{\ln(n + 1) - \ln n}{2} \cos \frac{\ln(n + 1) + \ln n}{2} = 2 \sin \frac{\ln \frac{n+1}{n}}{2} \cos \frac{\ln(n+1)n}{2}
\]
Since for any \( t \), \(-1 \leq \cos t \leq 1\), then
\[
-2 \sin \frac{\ln \frac{1 + \frac{1}{n}}{2}}{2} \leq 2 \sin \frac{\ln \frac{1 + \frac{1}{n}}{2}}{2} \cos \frac{\ln(n+1)n}{2} \leq 2 \sin \frac{\ln \frac{1 + \frac{1}{n}}{2}}{2}
\]
We have
\[
\lim \sin \frac{\ln \frac{1 + \frac{1}{n}}{2}}{2} = \sin \lim \left[ \frac{\ln \left( 1 + \lim \frac{1}{n} \right)}{2} \right] = \sin \frac{\ln 1}{2} = 0.
\]
Therefore, by the squeeze theorem,
\[
\lim e_n = \lim 2 \sin \frac{\ln \left( 1 + \frac{1}{n} \right)}{2} \cos \frac{\ln(n+1)n}{2} = 0.
\]
2. a) Give an example of sequences \( \{s_n\}, \{t_n\} \) such that \( \lim s_n = 0 \), \( \lim t_n = 0 \) and \( \lim \frac{s_n}{t_n} = 100 \).
b) Give an example of sequences \( \{s_n\}, \{t_n\} \) such that \( \lim s_n = 0 \), \( \lim t_n = 0 \) and \( \lim \frac{s_n}{t_n} = 0 \).
c) Give an example of sequences \( \{s_n\}, \{t_n\} \) such that \( \lim s_n = 0 \), \( \lim t_n = 0 \) and \( \lim \frac{s_n}{t_n} \) does not exist.

Solution.

1. \( s_n = \frac{100}{n}, \ t_n = \frac{1}{n} \)
2. \( s_n = \frac{1}{n^2}, \ t_n = \frac{1}{n} \)
3. If we take \( s_n = 0.2^n \) and \( t_n = 0.1^n \), then
\[
\lim \frac{s_n}{t_n} = \lim \left( \frac{0.2}{0.1} \right)^n = \lim 2^n
\]

Does not exist.
3. True or false?

a) If \( \{s_n\} \) and \( \{t_n\} \) are divergent sequences, then the sequence \( \{s_n + t_n\} \) is divergent as well.

b) If \( \{s_n\} \) and \( \{t_n\} \) are divergent sequences, then the sequence \( \{s_n \cdot t_n\} \) is divergent as well.

c) If \( \{s_n\} \) converges and \( \{t_n\} \) diverges, then the sequence \( \{s_n + t_n\} \) diverges.

d) If the sequence \( s_1, s_2, s_3, \ldots \) converges, then the sequence \( s_2, s_4, s_6, \ldots \) converges as well.

e) If \( \lim s_n t_n = 0 \), then \( \lim s_n = 0 \) or \( \lim t_n = 0 \).

Solution.

a) False. Take \( s_n = n \), \( t_n = -n \). Then \( \{s_n\} \), \( \{t_n\} \) are divergent sequences. On the other hand, \( s_n + t_n = 0 \). Hence \( \{s_n + t_n\} \) converges.

b) False. Take

\[
\begin{align*}
s_n &: 1, 0, 1, 0, \\
t_n &: 0, 1, 0, 1, 
\end{align*}
\]

Both \( \{s_n\} \) and \( \{t_n\} \) are divergent sequences, but \( s_n \cdot t_n = 0 \) for all \( n \). Hence, \( \{s_n \cdot t_n\} \) converges.

c) True. We will use a proof by contradiction. Suppose that sequence \( \{u_n = s_n + t_n\} \) converges. Then \( t_n = u_n - s_n \) and, therefore, \( \{t_n\} \) being a difference of two convergent sequences, should converge as well. But it contradicts to the initial condition that \( \{t_n\} \) is a divergent sequence. Hence, \( \{u_n\} \) must be divergent.

d) True. Let \( L \) be the limit of sequence \( s_1, s_2, s_3, \ldots \). We will show that the sequence \( s_2, s_4, s_6, \ldots \) converges to the same number.

Let \( \epsilon > 0 \). Then, since \( \lim s_n = 0 \), we can find \( n_0 \) (depending on \( \epsilon \)) such that for all \( n > n_0 \) the inequality \( |s_n - L| < \epsilon \) holds. Notice now that if \( n > n_0 \), then \( 2n > n_0 \). Hence, \( |s_{2n} - L| < \epsilon \). That means exactly that \( \lim s_{2n} = L \).

e) Let \( \{s_n\} \) and \( \{t_n\} \) be as in part b). Then \( \lim s_n t_n = \lim 0 = 0 \), but \( \lim s_n \neq 0 \) and \( \lim t_n \neq 0 \). In fact, these limits do not exist at all.

4. A student lives 0.6 miles away from campus and 0.4 miles away from a pub. Once, at noon, he went to class\(^1\), but changed his mind after ten minutes of walking and decided to go to the bar instead. Five minutes later, he changed his mind back and decided to go to campus again. After walking for two and a half minutes, he turned back to the bar. He felt guilty a minute and 15 seconds later and decided to go to class again, and so on...

Where will the student be at 12:20pm if he walks at a speed of 3 miles per hour?

\(^1\)Sources say Calc II.
Solution. First, we introduce the (one-dimensional) coordinate system, which has student’s house as it’s origin and its axis goes in the direction of the campus. Now, let $x_n$ be the coordinate of the student at the moment, when he changes the direction of this movement for the $n$-th time. Then

\[
x_1 = 3 \text{mph} \cdot \frac{1}{6} \text{hr} \\
x_2 = 3 \text{mph} \cdot \frac{1}{6} \text{hr} - 3 \text{mph} \cdot \frac{1}{12} \text{hr} \\
x_3 = 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{12} + 3 \cdot \frac{1}{24} \\
x_4 = 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{12} + 3 \cdot \frac{1}{24} - 3 \cdot \frac{1}{48} \\
x_5 = 3 \cdot \frac{1}{6} - 3 \cdot \frac{1}{12} + 3 \cdot \frac{1}{24} - 3 \cdot \frac{1}{48} + 3 \cdot \frac{1}{96} \\
\vdots \\
x_n = 3 \cdot \frac{1}{6} \left( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{16} + \cdots + \frac{(-1)^{n-1}}{2^{n-1}} \right)
\]

What we would like to find is $\lim x_n$. To this end, recall that for any $a \neq 1$,

\[
1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.
\]

Hence, the formula for $x_n$ can be written as

\[
x_n = 3 \cdot \frac{1}{6} \cdot \frac{1 - (-\frac{1}{2})^{n+1}}{1 - (-\frac{1}{2})} = \frac{1 - (-\frac{1}{2})^{n+1}}{3}
\]

Then

\[
\lim x_n = \lim \frac{1 - (-\frac{1}{2})^{n+1}}{3} = \frac{1}{3}.
\]