Topic: limit of a sequence; basic properties of series.

Two useful facts

• For any sequence \( \{a_n\} \), which goes to infinity, \( \lim \left( 1 + \frac{1}{a_n} \right)^{a_n} = e \).
• For any sequence \( \{b_n\} \), which converges to zero, \( \lim \frac{\sin b_n}{b_n} = 1 \).

1. For each of the given sequences, find its limit.

   a) \( \lim \left( 1 + \frac{1}{2n} \right)^{3n} \)

   b) \( \lim \left( \frac{n+1}{n+2} \right)^{n+3} \)

   c) \( \lim \frac{\sin \frac{3}{4n}}{\sin \frac{1}{n}} \)

   d) \( \lim \frac{\sin \frac{1}{n}}{\sqrt{n^2+1}-1} \)

Series

Definition. Let \( \{a_n\} \) be a sequence. Then the expression of the form \( \sum_{n=1}^{\infty} a_n \) or, simply, \( a_1 + a_2 + \cdots + a_n + \ldots \) is called a series.

We attach a numerical value ("the sum") to a series \( \sum_{n=1}^{\infty} a_n \) by forming a the sequence of partial sums

\[ s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad s_n = a_1 + a_2 + \cdots + a_n, \ldots \]

and then finding its limit \( s = \lim s_n \). It may happen that the limit does not exist. In such a case, we just say that the given series diverges.

1. For each of the given series, find a formula for its \( n \)-th partial sum \( s_n \) and then find the sum of the series (if it exists).

   a) \( 1 + 0.1 + 0.01 + 0.001 + \ldots \)
b) \[4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\]

c) \[\sum_{n=1}^{\infty} \left( \frac{3^n + 2^n}{3^n} \right)\]

d) \[x + 2x^2 + 3x^3 + 4x^4 + \ldots\]

e) \[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n(n+2)} + \ldots\]