Worksheet 8

Topics: power series expansion, Taylor series.

1. For each of the given functions, find its power series representation and determine the values of $x$ for which this representation is valid.
   a) $f(x) = \frac{x}{2-3x}$

   b) $f(x) = \frac{1}{(1+x)^2}$

2. Compute $\frac{1}{13} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} + \cdots + \frac{1}{n \cdot 3^n} + \cdots$

   *Hint:* Start by considering the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$.

Let $f = f(x)$ be a function, which has derivatives of all orders in some interval containing the point $x = a$. The power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n + \cdots$$

is called the Taylor series of $f$ at point $a$. Within its interval of convergence, the sum of this series is equal to $f(x)$.

Below is the list of some most basic and useful Taylor series:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, interval of convergence: $(-\infty, \infty)$;
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, interval of convergence: $(-\infty, \infty)$;
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, interval of convergence: $(-\infty, \infty)$;
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, interval of convergence: $(-1, 1)$;
- $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, interval of convergence: $(-1, 1)$;
- $(1 + x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots$, interval of convergence: $(-1, 1)$;
2. Find the Taylor series for \( f = f(x) \) at the given point \( a \) and determine its interval of convergence.

   a) \( f(x) = \cos x, \ a = \pi. \)

   b) \( f(x) = \sqrt{1 + x}, \ a = 0. \)

   c) \( f(x) = x \sin 2x, \ a = 0. \)

   d) \( f(x) = x^3 - 2x + 3, \ a = 2. \)

   e) \( f(x) = \ln(10 + x), \ a = 0. \)

   f) \( f(x) = \cos^2 x, \ a = 0. \)