Worksheet 14

Topics: bases and dimension of a solution space; eigenvalues and eigenvectors.

1. Find the dimension and a basis the solution space of the given linear system.
\[
\begin{align*}
x_1 - 3x_2 - 10x_3 + 5x_4 &= 0 \\
x_1 + 4x_2 + 11x_3 - 2x_4 &= 0 \\
x_1 + 3x_2 + 8x_3 - x_4 &= 0
\end{align*}
\]

2. Find the eigenvalues and the associated eigenvectors of the given matrix.

a) \[
\begin{bmatrix}
 4 & -3 \\
 2 & -1
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
 9 & -10 \\
 2 & 0
\end{bmatrix}
\]

c) \[
\begin{bmatrix}
 1 & 0 & 3 \\
 0 & 2 & -2 \\
 0 & 0 & 3
\end{bmatrix}
\]
3. Find a 2-by-2 matrix, which has the eigenvalues 1 and -1 with the corresponding eigenvectors $(1, 0)$ and $(1, 1)$ respectively.

4. Let $A$ be a square matrix (of arbitrary size) such that $A^{100} = 0$. Show that zero is the only eigenvalue of $A$.

5. Let $A$ be an invertible square matrix (of arbitrary size). Prove that if $\lambda$ is an eigenvalue of $A$, then $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

6. Let $A$ and $P$ be square matrices of the same size and $P$ is invertible. Show that matrices $A$ and $PAP^{-1}$ have the same eigenvalues.