When we wish to reduce substances to their principles, we state that lines come from the short and long, and the plane from the broad and narrow, and body from the deep and shallow.

Aristotle
Metaphysics

Worksheet 2

The vector equation of the line \( l \) passing through the point \( A(x_0, y_0, z_0) \) in the direction of the vector \( \mathbf{v} = (a, b, c) \) is \( \mathbf{l}(t) = A + tv \), where the parameter \( t \) takes on all real values. In coordinate form, the equation is \((x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \) or

\[
\begin{align*}
x &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct
\end{align*}
\]

1. Write an equation of the line \( l \).

a) The line \( l \) passes through the points \( A(-1, 2, 3) \), \( B(2, 6, -2) \).

**Solution.** A direction vector of the line \( l \) is vector \( \overrightarrow{AB} \). We have

\[
\overrightarrow{AB} = B - A = (3, 4, -5).
\]

Thus,

\[
(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \quad \text{or} \quad (x, y, z) = (-1, 2, 3) + t(3, 4, -5),
\]

which is

\[
(x, y, z) = (-1 + 3t, 2 + 4t, 3 - 5t),
\]

is an equation of the line \( l \).

b) The line \( l \) passes through the point \( A(0, 1, -3) \) parallel to the line \( m \) determined by the equations

\[
\begin{align*}
x &= 3 + t \\
y &= 1 - t \\
z &= 2 + 3t
\end{align*}
\]

**Solution.** Vector \( \mathbf{v} = (1, -1, 3) \) is a direction vector of the line \( m \). Since \( m \) is parallel to \( l \), then \( \mathbf{v} \) is also a direction vector of \( l \). Therefore, \( l \) can be determined by the equation

\[
(x, y, z) = (0, 1, -3) + t(1, -1, 3) = (t, 1 - t, -3 + 3t).
\]
c) The line $l$ passes through the point $A(1,1,2)$ perpendicular to the $y$-axis.

**Solution.** Let $B$ be the point of intersection of the line $l$ and the $y$-axis (make a sketch). The vector $\overrightarrow{AB}$ is a direction vector of the line $l$. To find the coordinates of $\overrightarrow{AB}$, we need to know the coordinates of point $B$. Since $B$ lies on the $y$-axis, it has coordinates $(0,\lambda,0)$ for some number $\lambda$. Since $l$ is perpendicular to the $y$-axis, the vector $\overrightarrow{AB}$ is orthogonal to the vector $j = (0,1,0)$. It implies, in particular, that $\overrightarrow{AB} \cdot (0,1,0) = 0$. We have

\[
\overrightarrow{AB} = (0,\lambda,0) - (1,1,2) = (-1,\lambda - 1,-2)
\]

\[
\overrightarrow{AB} \cdot (0,1,0) = (-1,\lambda - 1,-2) \cdot (0,1,0) = \lambda - 1.
\]

Thus, $\lambda$ must be equal to 1. Then $\overrightarrow{AB} = (-1,0,-2)$ and an equation of the line $l$ has the form

\[
(x,y,z) = (1,1,2) + t(-1,0,-2)
\]

\[
(x,y,z) = (1 - t,1,2 - 2t).
\]
The equation of the plane \( \alpha \) passing through the point \( A(x_0, y_0, z_0) \) perpendicular to the vector \( \mathbf{n} = (A, B, C) \) is

\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.
\]

Vector \( \mathbf{n} \) is called a normal vector.

2. Write an equation of the plane \( \alpha \).

a) The plane \( \alpha \) passes through the point \( A(1, 0, -2) \) perpendicular to the line \( l(t) = (2t, 1 - t, 3t) \).

Solution. In order to find an equation of a plane, we need to know its normal vector. In our case, since the given line is perpendicular to the plane \( \alpha \), then a direction vector of this line can be taken as a normal vector of the plane. The vector \( \mathbf{v} = (2, -1, 3) \) is a direction vector of the line. Hence, an equation of the plane is

\[
2(x - 1) - 1(y - 0) + 3(z + 2) = 0.
\]

After simplification, it becomes

\[
2x - y + 3z + 4 = 0.
\]

b) The plane \( \alpha \) passes through the point \( A(0, 4, 1) \) parallel to the plane \( x - y + z = 100 \).

Solution. If two planes \( \alpha \) and \( \beta \) are parallel, then their normal vectors are parallel. In particular, if \( \mathbf{n} \) is a normal vector of \( \beta \), then \( \mathbf{n} \) can serve as a normal vector of \( \alpha \). In our case, a normal vector of the plane \( x - y + z = 100 \) is \( \mathbf{n} = (1, -1, 1) \) (take the coefficients standing by the variables \( x, y \) and \( z \)). Hence, the plane \( \alpha \) has an equation

\[
1 \cdot (x - 0) - 1 \cdot (y - 4) + 1 \cdot (z - 1) = 0.
\]

After simplification,

\[
x - y + z + 3 = 0.
\]

c) Find an equation of the plane containing the lines \( c_1(t) = (1 + 2t, 2 - 3t, 3) \) and \( c_2(t) = (-t, t + 3, 1 - 2t) \).

Solution. Since lines \( c_1 \) and \( c_2 \) lie in the plane \( \alpha \), their direction vectors \( \mathbf{u} = (2, -3, 0) \) and \( \mathbf{v} = (-1, 1, -2) \) are also contained in \( \alpha \). Hence, the cross product \( \mathbf{u} \times \mathbf{v} \) is a vector perpendicular to \( \alpha \). We compute

\[
\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & -3 & 0 \\
-1 & 1 & -2
\end{vmatrix} = (6 - 0)\mathbf{i} - (4 - 0)\mathbf{j} + (2 - 3)\mathbf{k} = (6, 4, -1).
\]
The point \( A = c_1(0) = (1, 2, 3) \) certainly lies in the plane. Hence, the plane \( \alpha \) has an equation

\[
6 \cdot (x - 1) + 4 \cdot (y - 2) - 1 \cdot (z - 3) = 0,
\]

which is

\[
6x + 4y - z - 11 = 0.
\]

**Question.** How would you solve the problem, if the given two lines were parallel?

2. Determine if points \( A(2, -1, -2), B(1, 2, 1), C(2, 3, 0), D(5, 0, -6) \) lie in the same plane.

**Solution 1.** *Idea:* we will find an equation of the plane \( \alpha \) passing through the points \( A, B \) and \( C \) (see example 1.3.11, p.51) and check if point \( D \) satisfies this equation as well.

The vector \( \overrightarrow{AB} \times \overrightarrow{AC} \) is normal to the plane \( \alpha \). We find

\[
\overrightarrow{AB} = (-1, 3, 3), \quad \overrightarrow{AC} = (0, 4, 2)
\]

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 3 & 3 \\
0 & 4 & 2
\end{vmatrix} = (6 - 12)i - (-2 - 0)j + (-4 - 0)k = (-6, 2, 4).
\]

Thus the plane \( \alpha \) has an equation

\[
-6 \cdot (x - 2) + 2 \cdot (y + 1) - 4 \cdot (z + 2) = 0.
\]

Now we check whether the coordinates of the point \( D \) satisfy this equation:

\[
-6 \cdot (5 - 2) + 2 \cdot (0 + 1) - 4 \cdot (-6 + 2) = -18 + 2 + 16 = 0.
\]

They do. Hence, \( D \) is contained in the same plane as \( A, B \) and \( C \).

**Solution 2** (for those who are familiar with linear algebra).

The points \( A, B, C \) and \( D \) lie in the same plane, if and only if the vectors \( \overrightarrow{AB}, \overrightarrow{AC} \) and \( \overrightarrow{AD} \) lie in the same plane. Three vectors lie in the same plane if and only if they are linearly dependent, and three vectors in a three-dimensional space are linearly dependent when the 3-by-3 determinant made up of the coordinates of these vectors is zero. We compute

\[
\begin{vmatrix}
-1 & 3 & 3 \\
0 & 4 & 2 \\
3 & 1 & -4
\end{vmatrix} = (-1) \cdot (-16 - 2) - 3 \cdot (0 - 6) + 3 \cdot (0 - 12) = 0.
\]

So all four points are contained in the same plane.