'You see ... how shall I put it? Are you a mathematician?'
'Yes.'
'Then you see ... imagine a plane, let us say this mirror.
You and I are on its surface. You see? There we are...'

Yevgeny Zamyatin

We

Let $f = f(x, y)$ be a function differentiable at $x = x_0, y = y_0$. The equation of the plane tangent to the graph of this function at the point $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$$

1. a) Find an equation of the plane tangent to the surface $z = \frac{x^2}{2} - xy + 1$ at the point $(2, 1, 1)$.

**Solution.** If $f(x, y) = \frac{x^2}{2} - xy + 1$, then

$$f(2, 1) = 1$$

$$\frac{\partial f}{\partial x}(2, 1) = \left(\frac{2x}{2} - y\right)_{(2,1)} = 1$$

$$\frac{\partial f}{\partial y}(2, 1) = (-x)_{(2,1)} = -2.$$ 

So the tangent plane at the point $(2, 1, 1)$ has an equation

$$z = 1 + 1 \cdot (x - 2) + (-2) \cdot (y - 1),$$

or, after simplification, $x - 2y - z + 1 = 0$.

b) Find an equation of the plane tangent to the surface $x + 2y - \ln z + 4 = 0$ at the point $(2, -3, 1)$.

**Solution.** First, we need to express $z$ as a function of $x$ and $y$ from the given equation. We have

$$\ln z = x + 2y + 4$$

$$z = e^{x+2y+4}.$$ 

If we take $f(x, y) = e^{x+2y+4}$, then

$$f(2, -3) = e^{2-6+4} = 1$$

$$\frac{\partial f}{\partial x}(2, -3) = \left(e^{x+2y+4}\right)_{(2,-3)} = 1$$

$$\frac{\partial f}{\partial y}(2, -3) = \left(2e^{x+2y+4}\right)_{(2,-3)} = 2.$$ 

The tangent plane at the point \((2, -3, 1)\) has an equation
\[
z = 1 + 1 \cdot (x - 2) + 2 \cdot (y - (-3)).
\]
After simplification, it becomes
\[
x + 2y - z + 5 = 0.
\]
2. Find all points on the surface \(z = x^2y + 3xy + 6x + 1\) where the tangent plane is
   a) horizontal; b) parallel to the plane \(-3x + 2y + z = 10\).

Solution. Recall that the plane given by an equation \(Ax + By + Cz + D = 0\) has the vector \(n = (A, B, C)\) as its normal vector. Then the vector
\[
n = \left( \frac{\partial f}{\partial x} \bigg|_{(x_0, y_0)}, \frac{\partial f}{\partial y} \bigg|_{(x_0, y_0)}, -1 \right)
\]
is a normal vector of the tangent plane
\[
z = f(x_0, y_0) + \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} \cdot (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} \cdot (y - y_0)
\]

a) A horizontal plane has a vertical normal vector. The \(x\)- and \(y\)-components of such a vector are both zero. Hence, according to (1), to find a horizontal tangent plane, we need to solve the system of equations
\[
\begin{cases}
\frac{\partial f}{\partial x} = 0 \\
\frac{\partial f}{\partial y} = 0
\end{cases}
\]
In our case, \(f(x, y) = x^2y + 3xy + 6x + 1\) and we obtain
\[
\begin{cases}
\frac{\partial f}{\partial x} = 2xy + 3y + 6 = 0 \\
\frac{\partial f}{\partial y} = x^2 + 3x = 0
\end{cases}
\]
The second equation implies \(x = 0\) or \(x = -3\). Then the first equation yields \(y = -2\) or \(y = 2\) respectively. So the planes tangent to the given surface are horizontal at the points \((0, -2, f(0, -2)) = (0, -2, 1)\) and \((-3, 2, f(-3, 2)) = (-3, 2, -17)\).
b) Two distinct planes are parallel if and only if their normal vectors are parallel, and two vectors are parallel if and only if one of them is a multiple of the other. Hence, to find a tangent plane parallel to the plane \(-3x + 2y + z = 10\), we need to find the points where the vector

\[
\mathbf{n} = \left( \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right)
\]

is a multiple of \(\mathbf{m} = (-3, 2, 1)\). Comparing the \(z\)-components, we conclude that \(\mathbf{n}\) must be equal to \((-1)\mathbf{m}\). It gives us the system of equations

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2xy + 3y + 6 = 3 \\
\frac{\partial f}{\partial x} &= x^2 + 3x = -2
\end{align*}
\]

Solving the second equation, we obtain \(x = -2\) or \(x = -1\). Then the first one yields \(y = 3\) or \(y = -3\) respectively. Thus the planes tangent to the given surface at the points \((-2, 3, f(-2, 3)) = (-2, 3, -17)\) and \((-1, -3, f(-1, -3)) = (-1, -3, 1)\) are parallel to \(-3x + 2y + z = 10\).
3. The inner surface of a teacup is a paraboloid $z = x^2 + y^2 - 8$. A wasp, sitting in the cup at the point $(3, 0, 1)$, takes off perpendicular to the cup’s surface and flies straight until it reaches the opposite point. How far does the wasp fly?

Solution. Let us restate the problem in geometric terms:

Given a paraboloid $z = x^2 + y^2 - 8$ and a line passing through the point $A(3, 0, 1)$ perpendicular to the paraboloid. Let $B$ be the other point where the line intersects the surface. Find the distance $|AB|$.

Step 1. We find a vector perpendicular to the surface at the point $A$.

As we have seen before\(^1\), the vector

$$
n = \left( \frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right)$$

is a normal vector of the plane tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0, z_0)$ and that is a vector perpendicular to the surface at this point.

We compute the partial derivatives of the function $f(x, y) = x^2 + y^2 - 8$ at the point $A(3, 0, 1)$:

$$\frac{\partial f}{\partial x}(3, 0) = 2x = 6$$
$$\frac{\partial f}{\partial y}(3, 0) = 2y = 0$$

Then a normal vector is $n = (6, 0, -1)$. This vector is perpendicular to the paraboloid at the point $A$.

\(^1\)See the solution of problem 2.
Step 2. We find an equation of the line passing through the point \((3,0,1)\) in the direction given by the vector \(\mathbf{n} = (6,0,-1)\).

An equation of the line passing through the point \((x_0,y_0,z_0)\) in the direction of the vector \(\mathbf{d} = (a,b,c)\) is

\[
(x,y,z) = (x_0,y_0,z_0) + (a,b,c)t.
\]

In our case \((x_0,y_0,z_0) = (3,0,1)\) and the direction is given by the vector \(\mathbf{n} = (6,0,-1)\). Hence, an equation of the line is

\[
(x,y,z) = (3,0,1) + (6,0,-1)t
\]

or, equivalently, in coordinates

\[
\begin{cases} 
  x = 3 + 6t \\
  y = 0 \\
  z = 1 - t 
\end{cases}
\]

Step 3. We find the coordinates of the point of intersection of the line and the paraboloid \(z = x^2 + y^2 - 8\) other than \(A(3,0,1)\).

To find the points of intersection of the line and the paraboloid, we need to solve the system of equations

\[
\begin{cases} 
  z = x^2 + y^2 - 8 \\
  x = 3 + 6t \\
  y = 0 \\
  z = 1 - t 
\end{cases}
\]

Plugging the expressions for \(x\), \(y\) and \(z\) from the last three equations into the first equation, we obtain

\[
(1 - t) = (3 + 6t)^2 + 0^2 - 8 = 36t^2 + 37t = 0.
\]

Thus, \(t = 0\) or \(t = -\frac{37}{36}\), which implies

\[
x = 3, \quad y = 0, \quad z = 1
\]

or

\[
x = 3 + 6 \cdot \left( -\frac{37}{36} \right), \quad y = 0, \quad z = 1 - \left( -\frac{37}{36} \right).
\]

The first solution is exactly point \(A\) and the second solution gives us another point \(B\).

Step 4. We find the distance \(|AB|\).

\[
|AB| = \sqrt{6^2 \cdot \left( -\frac{37}{36} \right)^2 + 0^2 + (-1)^2 \cdot \left( -\frac{37}{36} \right)^2} = \frac{37\sqrt{37}}{36}.
\]
The values of the linear
function
\[ z(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}_{(x_0, y_0)} \cdot (x - x_0) + \frac{\partial f}{\partial y}_{(x_0, y_0)} \cdot (y - y_0) \]
are close to the actual values of the function \( f = f(x, y) \) if \( x \) and \( y \) are close to \( x_0, y_0 \). This property can be used to perform quick approximate computations.

4. Use the linear approximation formula to estimate
a) \( \ln(\sqrt{1.04} + \sqrt{1.01} - 1) \); b) \( 1.04^{2.02} \).

Solution.

a) We let \( f(x, y) = \ln(\sqrt{x} + \sqrt{y} - 1) \), \( x_0 = 1 \), \( y_0 = 1 \) and compute
\[
\begin{align*}
f(x_0, y_0) &= \ln(\sqrt{x} + \sqrt{y} - 1) \bigg|_{(1,1)} = 0 \\
\frac{\partial f}{\partial x}_{(x_0, y_0)} &= \frac{1}{\sqrt{x} + \sqrt{y} - 1} \cdot \frac{1}{2\sqrt{x}} \bigg|_{(1,1)} = \frac{1}{2} \\
\frac{\partial f}{\partial y}_{(x_0, y_0)} &= \frac{1}{\sqrt{x} + \sqrt{y} - 1} \cdot \frac{1}{3\sqrt{y^2}} \bigg|_{(1,1)} = \frac{1}{3}
\end{align*}
\]
Then the linear approximation for \( f \) at the point \((1, 1)\) is
\[
z(x, y) = 0 + \frac{1}{2} \cdot (x - 1) + \frac{1}{3} \cdot (y - 1)
\]
and we obtain
\[
\ln(\sqrt{1.04} + \sqrt{1.01} - 1) \approx z(1.04, 1.01) = 0 + \frac{1}{2} \cdot (1.04 - 1) + \frac{1}{3} \cdot (1.01 - 1) = 0.0233...
\]

b) We let \( f(x, y) = x^y \), \( x_0 = 1 \), \( y_0 = 2 \) and compute
\[
\begin{align*}
f(x_0, y_0) &= x^y \bigg|_{(1,2)} = 1 \\
\frac{\partial f}{\partial x}_{(x_0, y_0)} &= yx^{y-1} \bigg|_{(1,2)} = 2 \\
\frac{\partial f}{\partial y}_{(x_0, y_0)} &= x^y \ln x \bigg|_{(1,2)} = 0
\end{align*}
\]
Then the linear approximation for \( f \) at the point \((1, 2)\) is
\[
z(x, y) = 1 + 2 \cdot (x - 1) + 0 \cdot (y - 2)
\]
and we obtain
\[
1.04^{2.02} \approx z(1.04, 2.02) = 1 + 2 \cdot (1.04 - 1) + 0 \cdot (2.02 - 2) = 1.08.
\]
5. Using the linear approximation formula, decide what makes a greater impact on the value of $f(x, y) = x^y$ at $x = y = 10$: increasing $x$ by 0.2 or increasing $y$ by 0.1.

**Solution.** We compute

$$f(10, 10) = 10^{10}$$

$$
\frac{\partial f}{\partial x} \bigg|_{(10, 10)} = yx^{y-1} \bigg|_{(10, 10)} = 10^{10}
$$

$$
\frac{\partial f}{\partial y} \bigg|_{(10, 10)} = x^y \ln x \bigg|_{(10, 10)} = 10^{10} \ln 10.
$$

So the linear approximation for $f$ at the point $(10, 10)$ is

$$z(x, y) = 10^{10} + 10^{10} \cdot (x - 10) + 10^{10} \ln 10 \cdot (y - 10).$$

Then we have

$$10^{10} - 10^{10} \approx z(10.2, 10) - 10^{10} = 10^{10} \cdot 0.2$$

$$10^{10} - 10^{10} \approx z(10, 10.1) - 10^{10} = 10^{10} \ln 10 \cdot 0.1$$

It remains to compare 0.2 and $\ln 10 \cdot 0.1$. Since $e = 2.71828...$, we have $e^2 < 9 < 10 = e^{\ln 10}$. Since $f(t) = e^t$ is a strictly increasing function, then the last inequality implies $2 < \ln 10$ and $0.2 < \ln 10 \cdot 0.1$. Therefore, increasing $y$ by 0.1 makes a greater impact.