Worksheet 5

Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) and \( g : \mathbb{R}^m \to \mathbb{R}^s \) be differentiable functions and \( x = (x_1, x_2, \ldots, x_n) \) be some point. The Chain Rule helps to compute the matrix of the partial derivatives of the composite function \((g \circ f) : \mathbb{R}^n \to \mathbb{R}^s:\)

Step 1. Compute the \( m \)-by-\( n \) matrix \( Df(x) \).

Step 2. Compute the \( s \)-by-\( m \) matrix \( Dg \) at the point \( f(x) \).

Step 3. Compute the product \( Dg(f(x)) \cdot Df(x) \). This matrix is equal to \( D(g \circ f) \) at the point \( x \).

1. Let \( f(x, y) = \left( x + y, \sin(xy) \right) \) and \( g(x, y) = \left( \sin(x + y), e^y \cos(x) \right) \). Using the Chain Rule, find \( D(g \circ f)(1, \pi) \).

\[
\begin{bmatrix}
1 & 1 \\
\sin & 1 + \sin
\end{bmatrix}
\]

2. Let \( v(x, y, z) = \frac{xy}{x+y} \) and \( u(x, y) = (y, x^2 + y^2, x^2 - y^2) \). Using the Chain Rule, evaluate the partial derivative \( \frac{\partial (v \circ u)}{\partial y} \bigg|_{(-1,1)} \).

\[
\begin{bmatrix}
1 & 1 \\
\sin & 1 + \sin
\end{bmatrix}
\]

Answer:
3. The temperature in the room at point \((x, y, z)\) is given by 
\[
T(x, y, z) = 10e^{-(x^2+y^2+z^2)}+20z.
\]
A bee is flying up along the path \(c(t) = \left( t \cos \pi t, t \sin \pi t, t^2 \right)\). Is it getting warmer or colder for the bee at the point \((-3, 0, 9)\)?

4. Let \(f(x, y) = (x+y, x-y)\). It is given that 
\[
\frac{\partial (g \circ f)}{\partial x} \bigg|_{(1,0)} = 5 \quad \text{and} \quad \frac{\partial (g \circ f)}{\partial y} \bigg|_{(1,0)} = -3
\]
for some function \(g : \mathbb{R}^2 \to \mathbb{R}\). Find the partial derivative 
\[
\frac{\partial g}{\partial x} \bigg|_{(1,1)}.
\]