Worksheet 7

Geometric interpretation of a double integral. The double integral \( \int \int_D f(x, y) \, dA \) of a non-negative function \( f = f(x, y) \) is equal to the volume of a solid lying over the region \( D \) under the surface \( z = f(x, y) \).

Physical interpretation of a double integral. Let \( D \) be a thin plate contained in the \( xy \)-plane. If \( f = f(x, y) \) is a function that gives the area density of the plate at point \( (x, y) \), then the double integral \( \int \int_D f(x, y) \, dA \) is equal to the total mass of the plate.

1. Find the volume of the solid \( W \).

   a) The solid \( W \) is the solid bounded by the paraboloid \( z = x^2 + y^2 \) and the planes \( x = 0, \ y = 0, \ z = 0, \ x + y = 1 \).

   b) The solid \( W \) is the solid bounded by the surface \( x^2 + y^2 = 2y \) and the planes \( z = y, \ z = 0 \).

      Hint: Completing the square in the first equation might be helpful.

2. Fill in the blanks.

   \[
   \int_{2^{x-1}}^{1} \int_{0}^{...} f(x, y) \, dy \, dx = \int_{...}^{0} \int_{y}^{...} f(x, y) \, dx \, dy
   \]
3. A plate has the shape of the square $0 \leq x \leq 1$, $0 \leq y \leq 1$. The area density of the material at point $(x, y)$ is equal to $\rho(x, y) = \frac{y}{(1+y^2)^2}$ units. We cut the plate into two pieces along the curve $y = \sqrt{x}$. Which of these pieces is heavier?

4. Sketch the domain of integration and evaluate the integral.
   
   a) $\int \int \int_W yz \, dx \, dy \, dz$, where $W$ is the solid bounded by the surfaces $x + y = 1$, $x = 0$, $y = 0$, $z = 0$, $z = 1$.
   
   Answer: $\frac{11}{12}$.

   b) $\int \int \int_W \frac{x}{1+z} \, dx \, dy \, dz$, where $W$ is the solid determined by the inequalities $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$, $0 \leq x \leq 1$.