Worksheet 7

Geometric interpretation of a double integral. The double integral \( \int \int_D f(x, y) \, dA \) of a non-negative function \( f = f(x, y) \) is equal to the volume of a solid lying over the region \( D \) under the surface \( z = f(x, y) \).

Physical interpretation of a double integral. Let \( D \) be a thin plate contained in the \( xy \)-plane. If \( f = f(x, y) \) is a function that gives the area density of the plate at point \( (x, y) \), then the double integral \( \int \int_D f(x, y) \, dA \) is equal to the total mass of the plate.

1. Find the volume of the solid \( W \).

   a) The solid \( W \) is the solid bounded by the paraboloid \( z = x^2 + y^2 \) and the planes \( x = 0, \ y = 0, \ z = 0, \ x + y = 1 \).

   Solution. That is how the given solid looks like:

   ![Graph of the solid]

   It is the solid lying under the graph of the function \( z(x, y) = x^2 + y^2 \) over the triangle

   \[ 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x \]

   Thus the volume is equal to

   \[
   \int \int_D x^2 + y^2 \, dA = \int_0^1 \int_0^{1-x} x^2 + y^2 \, dy \, dx = \int_0^1 \left( x^2 y + \frac{y^3}{3} \right) \bigg|_{y=0}^{y=1-x} \, dx
   \]

   \[
   = \int_0^1 x^2(1 - x) + \frac{(1 - x)^3}{3} \, dx = \frac{1}{6}.
   \]
b) The solid $W$ is the solid bounded by the surface $x^2 + y^2 = 2y$ and the planes $z = y, z = 0$.

Hint: Completing the square in the first equation might be helpful.

In order to see what surface is defined by the equation $x^2 + y^2 = 2y$, we complete the square (we are reducing it to the canonical form):

\[
x^2 + (y^2 - 2y) = 0 \\
x^2 + (y^2 - 2y + 1) = 1 \\
x^2 + (y - 1)^2 = 1
\]

The equation $x^2 + (y - 1)^2 = 1$ defines a circular cylinder of radius 1 whose axis is vertical and passes through the point $(0, 1)$ on the $xy$-plane. The planes $z = y$ and $z = 0$ cut a certain piece of this cylinder off. So the given solid looks as follows:

![Diagram of the solid W](image)

It can be described by the inequalities

\[-1 \leq x \leq 1, \quad 1 - \sqrt{1 - x^2} \leq y \leq 1 + \sqrt{1 - x^2}\]

To find the volume, we evaluate the integral

\[
\int \int_D y \, dA = \int \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} y \, dy \, dx = \int_{-1}^{1} \frac{y^2}{2} \bigg|_{y=1-\sqrt{1-x^2}}^{y=1+\sqrt{1-x^2}} dx = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx.
\]

To compute the last integral, one can make the standard substitution $x = \sin \theta$ or use the geometric meaning of a definite integral. Namely, $\int_{-1}^{1} \sqrt{1-x^2} \, dx$ is equal to the area under the curve $y = \sqrt{1-x^2}$ over the interval $[-1, 1]$. The curve $y = \sqrt{1-x^2}$ is the upper half-circle of radius 1. Thus the area is equal to $\frac{\pi}{2}$. Then

\[
\int \int_D y \, dA = \pi.
\]
2. Fill in the blanks.

\[
\int_{\frac{1}{2}}^{1} \int_{2^{x-1}}^{\cdots} f(x, y) \, dy \, dx = \int_{0}^{\cdots} \int_{y}^{\cdots} f(x, y) \, dx \, dy
\]

**Solution.** From the given equality, we deduce that the domain of integration \( D \) is bounded by the curves \( y = 2^{x} - 1 \), \( x = y \). That is how it looks like:

Thus

\[
\int_{0}^{1} \int_{2^{x-1}}^{x} f(x, y) \, dy \, dx = \int_{0}^{1} \frac{\ln(y+1)}{\ln 2} \int_{y}^{\cdots} f(x, y) \, dx \, dy.
\]

3. A plate has the shape of the square \( 0 \leq x \leq 1 \), \( 0 \leq y \leq 1 \). The area density of the material at point \( (x, y) \) is equal to \( \rho(x, y) = \frac{y}{(1+y^2)^2} \) units. We cut the plate into two pieces along the curve \( y = \sqrt{x} \). Which of these pieces is heavier?

**Solution.** The mass of the upper part is given by the integral

\[
\int_{\text{upper region}} \int \frac{y}{(1+y^2)^2} \, dA
\]

The upper region is determined by the inequalities

\[
0 \leq x \leq 1, \quad \sqrt{x} \leq y \leq 1.
\]
We compute
\[
\int \int_{\text{upper region}} \frac{y}{(1 + y^2)^2} \, dA = \int_0^1 \left( \int_0^1 \frac{y}{(1 + y^2)^2} \, dy \right) \, dx = \frac{1}{2} \int_0^1 \left( \int_0^2 \frac{1}{u^2} \, du \right) \, dx
\]
\[
= \frac{1}{2} \int_0^1 \left( -\frac{1}{u} \right)_{u=1+x} \, dx = \frac{1}{2} \int_0^1 \left( \frac{1}{1+x} - \frac{1}{2} \right) \, dx = \frac{1}{2} \ln 2 - \frac{1}{4}. 
\]

The mass of the lower part is equal to the total mass of the plate with the mass of the upper part subtracted. We compute

Total mass = \[
\int \int_{0}^{1} \int_{0}^{1} \frac{y}{(1 + y^2)^2} \, dy \, dx = \int_0^1 \left( \int_0^1 \left( -\frac{1}{u} \right)_{u=1+x} \, du \right) \, dx = \int_0^1 \left( \frac{1}{4} \right) \, dx = \frac{1}{4}. 
\]

Hence, the mass of the lower part is equal to

\[
\frac{1}{4} - \left( \frac{1}{2} \ln 2 - \frac{1}{4} \right) = \frac{1}{2} - \frac{1}{2} \ln 2 \text{ units}. 
\]

We observe that \( \frac{1}{2} - \frac{1}{2} \ln 2 > \frac{1}{2} \ln 2 - \frac{1}{4} \). Thus the lower part is heavier.

**Triple integral**

4. Sketch the domain of integration and evaluate the integral.

a) \( \int \int \int_{W} yz \, dx \, dy \, dz \), where \( W \) is the solid bounded by the surfaces \( x + y = 1, \ x = 0, \ y = 0, z = 0, \ z = 1. \)

**Solution.** The solid \( W \) is a prism:
\[
\int \int \int_W yz \, dx \, dy \, dz = \int_0^1 \int_0^1 \int_0^{1-y} yz \, dx \, dy \, dz = \int_0^1 \int_0^{1-y} xyz \bigg|_{x=0}^{x=1-y} \, dy \, dz
\]

\[
= \int_0^1 \int_0^1 yz - y^2z \, dy \, dz = \int_0^1 \frac{y^2}{2}z - \frac{y^3}{3} \bigg|_{y=0}^{y=1} \, dz
\]

\[
= \int_0^1 \frac{z}{2} \, dz = \frac{z^2}{4} - \frac{z^2}{6} \bigg|_{z=0}^{z=1} = \frac{1}{12}.
\]

b) \(\int \int \int_W \frac{x}{1+z} \, dx \, dy \, dz\), where \(W\) is the solid determined by the inequalities \(x^2 + y^2 \leq 1\), \(0 \leq z \leq 1\), \(0 \leq x \leq 1\).

**Solution.** The region \(W\) is a half of a solid cylinder:

\[
\int \int \int_W \frac{x}{1+z} \, dx \, dy \, dz = \int_0^1 \int_{-1}^0 \int_0^{\sqrt{1-y^2}} \frac{x}{1+z} \, dx \, dy \, dz = \int_0^1 \int_{-1}^0 \frac{x^2}{2(1+z)} \bigg|_{x=0}^{x=\sqrt{1-y^2}} \, dy \, dz
\]

\[
= \int_0^1 \int_{-1}^0 \frac{1-y^2}{2(1+z)} \, dy \, dz = \int_0^1 \frac{y - \frac{y^3}{3}}{2(1+z)} \bigg|_{y=-1}^{y=1} \, dz
\]

\[
= \int_0^1 \frac{4}{3} \, dz = \frac{2}{3} \ln(1+z) \bigg|_{z=0}^{z=1} = \frac{2}{3} \ln 2.
\]