Worksheet 10

Review of polar coordinates

1. Find the polar coordinates of the points $A(-1, \sqrt{3})$, $B(2, 0)$ and the midpoint $C$ of the line segment $AB$.

\[
\begin{align*}
\rho_A &= 2, \quad \theta_A = \frac{2\pi}{3}, \\
\rho_B &= 2, \quad \theta_B = \frac{\pi}{4}, \\
\rho_C &= 1, \quad \theta_C = \frac{\pi}{3}.
\end{align*}
\]

2. The points $A$ and $B$ have polar coordinates \((1, \frac{5\pi}{12})\) and \((2, \frac{\pi}{12})\) respectively. Find the distance \(|AB|\).

\[
\sqrt{3}.
\]

3. Sketch the plane curves given by the equations in the polar coordinates.
   
   a) $\theta = 1$;
   
   b) $\rho = 2 \sin \theta$;
   
   c) $\rho = \frac{2}{\sin \theta}$;
   
   d) $\rho = \theta$;
   
   e) $\rho^2 \sin 2\theta = 1$.

4. Sketch the regions given in the polar coordinates.

   a) $1 < \rho \leq 2$;
   
   b) $\rho \leq 2, \quad \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$;
   
   c) $\frac{\pi}{4} < \theta < \frac{5\pi}{4}$;
   
   d) $-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad 0 \leq \rho \leq \frac{\cos \theta}{\sin^2 \theta}$.

5. Describe the regions in polar coordinates.
Changing variables in a multiple integral

A change of variables in a double integral \( \int \int_D f(x, y) \, dx \, dy \), can be done as follows:

Step 1. Choose two continuously differentiable functions \( x = x(u, v), \ y = y(u, v) \) (\( u \) and \( v \) will be our new variables in the integral). The choice of these functions depends on the problem.

*Example:* Usually, changing to polar coordinates (via \( x = \rho \cos \theta, \ y = \rho \sin \theta \) is worth trying when 1) the region of integration \( D \) looks like a part of a disk or has a nice description in polar coordinates; 2) the function \( f = f(x, y) \) has some sort of a rotational symmetry (a good sign of it is the presence of the term \( x^2 + y^2 \) or similar).

Step 2. Describe the region \( D \) using the new variables \( u \) and \( v \). In other words, find the region \( D^* \) in the \( uv \)-plane such that \( x = x(u, v), \ y = y(u, v) \) would map \( D^* \) to \( D \).

Step 3. Compute the Jacobian determinant \( \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \). We will denote this determinant by \( \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \).

*Example:* For polar coordinates, \( \left| \frac{\partial (x, y)}{\partial (\rho, \theta)} \right| = \rho \).

Step 4. Evaluate the double integral \( \int \int_{D^*} f(x(u, v), y(u, v)) \cdot \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, du \, dv \).

6. Changing to polar coordinates, evaluate the integral \( \int \int_D \ln(1 + x^2 + y^2) \, dA \), where \( D \) is the quarter of the unit disk lying in the first quadrant \( (x \geq 0, \ y \geq 0) \).

\[ \frac{\pi}{4} \left(2 \ln 2 - 1\right) \]

6. Changing to polar coordinates, evaluate the integral \( \int \int_D 2xy \, dx \, dy \), where \( D \) is the region lying between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 2 \) in the second quadrant.

\[ \frac{5}{6} \]

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