Worksheet 9.5

Changing variables in a multiple integral

A change of variables in a triple integral \( \int \int \int_W f(x, y, z) \, dx \, dy \, dz \), can be done as follows:

Step 1. Choose three continuously differentiable functions \( x = x(u, v, w), \) \( y = y(u, v, w), \) \( z = z(u, v, w) \) (\( u, v \) and \( w \) will be our new variables in the integral). The choice of these functions depends on the problem.

Example: Usually, changing to spherical coordinates (via \( x = \rho \cos \theta \sin \phi, \) \( y = \rho \sin \theta \sin \phi, \) \( z = \rho \cos \phi \)) or cylindrical coordinates (via \( x = r \cos \theta, \) \( y = r \sin \theta, \) \( z = z \)) is worth trying when 1) the region of integration \( D \) looks like a part of a ball or a solid cylinder, or it has a 'nice' description in spherical or cylindrical coordinates; 2) the function \( f = f(x, y, z) \) has some sort of a 'rotational symmetry' (a good sign is the presence of the term \( x^2 + y^2, \) \( x^2 + y^2 + z^2 \) or similar).

Step 2. Describe the region \( W \) using the new variables \( u, v \) and \( w \). In other words, find the region \( W^* \) in the \( uvw \)-space such that \( x = x(u, v, w), \) \( y = y(u, v, w), \) \( z = z(u, v, w) \) would map \( W^* \) to \( W \).

Step 3. Compute the Jacobian determinant

\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix}
\]

We will denote this determinant

\[
\begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (u, v, w)}
\end{vmatrix}
\]

Example: For cylindrical coordinates, \( \begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (r, \theta, z)}
\end{vmatrix} = r \); for spherical coordinates, \( \begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (\rho, \theta, \phi)}
\end{vmatrix} = \rho^2 \sin \phi \).

Step 4. Evaluate the triple integral \( \int \int \int_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (u, v, w)}
\end{vmatrix} \, du \, dv \, dw \).

1. Changing to spherical coordinates, evaluate the integral \( \int \int \int_W (x^2 + y^2) \, dV \), where \( W \) is the region determined by the inequality \( 1 \leq x^2 + y^2 + z^2 \leq 4 \).
2. Changing to cylindrical coordinates, evaluate the integral \( \int \int \int_P \frac{1}{\sqrt{1+x^2+y^2}} dV \), where \( P \) is the region bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 1 \).

\[ (1 - \xi^2) \mu^2 \cdot \text{Vol} \]

3. Find the volume of the solid bounded by the sphere \( x^2 + y^2 + z^2 = \frac{3}{2} \) and the cone \( x^2 + y^2 - z^2 = 0 \).

\[ \pi \left( \sqrt{3} - \tau \right) \frac{\sqrt{3}}{2} \cdot \text{Vol} \]

4. Let \( D \) be the region in the first quadrant of the \( xy \)-plane bounded by the curves \( xy = 1 \), \( xy = 2 \), \( y = x \), \( y = 4x \). Find the area of \( D \).

\( \text{Hint: Make an appropriate change of variables in the area integral.} \)