Worksheet 11\frac{1}{2}

Changing variables in a multiple integral

1. (Viviani’s problem) Find the volume of the region bounded by the sphere \(x^2+y^2+z^2 = a^2\) and the cylinder \(x^2 + y^2 = ax\).

\textit{Hint:} In cylindrical coordinates \((r, \theta, z)\) the cylinder \(x^2 + y^2 = ax\) is given by the equation \(r = a \cos \theta\).

\[ \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{a \cos \theta}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} r \, dz \, dr \, d\theta \]

Parametrizing surfaces

To parametrize a surface \(S\) in the 3-dimensional Euclidean space means to find a function \(T = T(u,v)\) defined in some region \(D\) on the \(uv\)-plane, such that the image \(T(D)\) is \(S\).

\textit{Example:} Let \(D\) be the rectangle \([0, 2\pi] \times [0, \pi/2]\) on the \(\theta\phi\)-plane. We define \(T(\theta, \phi) = (2 \cos \theta \sin \phi, 2 \sin \theta \cos \phi, 2 \sin \phi)\). Then \(T : D \to \mathbb{R}^3\) is a parametrization of the upper hemisphere of radius 2.

2. Sketch the surface and find a parametrization for it.

a) A parabolic surface: \(z = x^2 + y^2, \; z \leq 4\).

b) A “cap” of the sphere: \(x^2 + y^2 + z^2 = 9, \; z \geq 2\).

c) A cylindrical surface: \(y^2 + z^2 = 1, \; |x| \leq 1\).

d) The part of the paraboloid \(y = 1 - x^2 - z^2\) with \(y \geq 0\).

e) Triangle with the vertices \((0, 0, 2), (1, 0, 0), (0, 2, 0)\).

\footnote{Vincenzo Viviani (1622-1703) was an Italian mathematician and scientist.}
2. Sketch the surface.

a) \( \mathbf{T}(u, v) = (u, v, u), \ 0 \leq u \leq 1, \ 0 \leq v \leq 1. \)

b) \( \mathbf{T}(u, v) = (u, 1, u^2), \ -1 \leq u \leq 1, \ 2 \leq v \leq 3. \)

c) \( \mathbf{T}(u, v) = (v, \cos u, \sin u), \ 0 \leq u \leq \pi, \ |v| \leq 3. \)

d) \( \mathbf{T}(u, v) = (v \cos u, v \sin u, v), \ 0 \leq u \leq 2\pi, \ -3 \leq v \leq 0. \)

e) \( \mathbf{T}(u, v) = (3 \cos u \cos v + 1, 3 \sin u \cos v - 1, 3 \sin v), \ 0 \leq u \leq 2\pi, \ 0 \leq v \leq \pi. \)