Worksheet 11

Changing variables in a multiple integral

A change of variables in a triple integral \( \int \int \int_W f(x, y, z) \, dx \, dy \, dz \), can be done as follows:

Step 1. Choose three continuously differentiable functions \( x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \) (\( u, v \) and \( w \) will be our new variables in the integral). The choice of these functions depends on the problem.

Example: Usually, changing to spherical coordinates (via \( x = \rho \cos \theta \sin \phi, y = \rho \sin \theta \sin \phi, z = \rho \cos \phi \)) or cylindrical coordinates (via \( x = r \cos \theta, y = r \sin \theta, z = z \)) is worth trying when 1) the region of integration \( D \) looks like a part of a ball or a solid cylinder, or it has a ‘nice’ description in spherical or cylindrical coordinates; 2) the function \( f = f(x, y, z) \) has some sort of a ‘rotational symmetry’ (a good sign is the presence of the term \( x^2 + y^2, x^2 + y^2 + z^2 \) or similar).

Step 2. Describe the region \( W \) using the new variables \( u, v \) and \( w \). In other words, find the region \( W^* \) in the \( uvw \)-space such that \( x = x(u, v, w), y = y(u, v, w), z = z(u, v, w) \) would map \( W^* \) to \( W \).

Step 3. Compute the Jacobian determinant

\[
\left| \begin{array}{ccc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{array} \right|
\]

We will denote this determinant

\[
\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|
\]

Example: For cylindrical coordinates, \( \left| \frac{\partial(x,y,z)}{\partial(r,\theta,z)} \right| = r \); for spherical coordinates, \( \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = \rho^2 \sin \phi \).

Step 4. Evaluate the triple integral \( \int \int \int_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, du \, dv \, dw \).
1. Changing to spherical coordinates, evaluate the integral \( \int \int \int_W (x^2 + y^2) \, dV \), where \( W \) is the region determined by the inequality \( 1 \leq x^2 + y^2 + z^2 \leq 4 \).

**Solution.**

**Step 1.** We change to spherical coordinates \((\rho, \theta, \phi)\) by making the substitution
\[
x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi.
\]

**Step 2.** The equations \( x^2 + y^2 + z^2 = 1 \) and \( x^2 + y^2 + z^2 = 4 \) define spheres centered at the origin of radii 1 and 2 respectively. Thus the region \( W \) enclosed between these spheres can be defined by the inequalities
\[
0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi, \quad 1 \leq \rho \leq 2.
\]

**Step 3.** The Jacobian determinant, when changing from Cartesian to spherical coordinates, is equal to
\[
\left| \frac{\partial (x, y, z)}{\partial (\rho, \theta, \phi)} \right| = \rho^2 \sin \phi.
\]

**Step 4.** We compute
\[
\int \int \int_W (x^2 + y^2) \, dV = \int_0^{2\pi} \int_0^\pi \int_1^2 \left( (\rho \cos \theta \sin \phi)^2 + (\rho \sin \theta \sin \phi)^2 \right) \rho^2 \sin \phi \, d\rho d\phi d\theta
\]
\[
= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^4 \sin^3 \phi \, d\rho d\phi d\theta
\]
\[
= \frac{31}{5} \int_0^\pi \sin^2 \phi \cdot \sin \phi \, d\phi d\theta = \frac{31}{5} \int_0^\pi (1 - \cos^2 \phi) \cdot \sin \phi \, d\phi d\theta
\]
\[
= -\frac{31}{5} \int_0^{2\pi} \int_0^1 (1 - u^2) \, du d\theta = -\frac{31}{5} \int_0^{2\pi} \left( u - \frac{u^3}{3} \right) \left|_{u=-1}^{u=1} \right. d\theta
\]
\[
= \frac{124}{15} \int_0^{2\pi} 1 \, d\theta = \frac{248}{15} \pi.
\]
2. Changing to cylindrical coordinates, evaluate the integral \( \int \int \int_P \frac{1}{\sqrt{1+x^2+y^2}} \, dV \), where \( P \) is the region bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 1 \).

**Solution.**

Step 1. As suggested, we will set up the integral in cylindrical coordinates \((r, \theta, z)\). That means that we are making the substitution

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \]

Step 2. This is how the region \( P \) looks like:

Due to rotational symmetry, the \( \theta \)-coordinate varies within \( P \) from 0 to \( 2\pi \).

We can notice that \( z \) and \( r \) depend on each other. In cylindrical coordinates the equation \( z = x^2 + y^2 \) becomes \( z = r^2 \). Thus, \( r \) varies from 0 to \( \sqrt{z} \) as \( z \) goes from 0 to 1.

So \( P \) is determined by the inequalities

\[ 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1, \quad 0 \leq r \leq \sqrt{z}. \]

Step 3. The Jacobian determinant, when changing from Cartesian to cylindrical coordinates, is equal to

\[ \left| \begin{array}{ccc} \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \end{array} \right| = r. \]
Step 4. We compute

\[
\int \int \int_P \frac{1}{\sqrt{1 + x^2 + y^2}} \, dV = \int \int \int_0^1 \frac{1}{\sqrt{1 + r^2}} \cdot r \, dz \, d\theta = \int_{\text{Jacobian}} \frac{u}{2} \, dz \, d\theta = \frac{1}{2} \int 1 + z \, dz \, d\theta
\]

\[
= \int \int \int \frac{1}{2\sqrt{u}} \, du \, dz \, d\theta = \int \frac{1}{2} \sqrt{1 + z} \, dz \, d\theta
\]

\[
= \int \int \int (\sqrt{1 + z} - 1) \, dz \, d\theta = \frac{2}{3} \int (1 + z) \, dz \, d\theta
\]

\[
= \frac{2}{3} \int (1 + z) \, d\theta = \frac{4}{3} \pi (2\sqrt{2} - 1).
\]

3. Find the volume of the solid bounded by the sphere \(x^2 + y^2 + z^2 = \frac{3}{2}\) and the cone \(x^2 + y^2 - z^2 = 0\).

Solution. This is how the given solid (we will denote it by \(W\)) looks like (on the right):

Its volume is equal to the triple integral \(\int \int \int_W 1 \, dV\). We are going to set up and evaluate this integral in spherical coordinates.

Step 1. We change to spherical coordinates \((\rho, \theta, \phi)\) by making the substitution

\[
x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi.
\]

In our case, since the integrand is a constant function, we do not need to plug anything in it.
Step 2. By symmetry, we can compute the volume of the upper part of the solid \( W \) and multiply the result by two. Consider the cross-section of \( W \) by the vertical plane \( y = 0 \):

As we can see, within \( W \) the \( \theta \)-coordinate varies from 0 to \( 2\pi \) and \( \rho \) goes from 0 to \( \sqrt{\frac{3}{2}} \) (the radius of the sphere).

To find the bounds for \( \phi \), let us recall the Cartesian equations of the given surfaces. The solution of the system of equations

\[
\begin{cases}
   x^2 + y^2 + z^2 = \frac{3}{2} \\
   x^2 + y^2 - z^2 = 0
\end{cases}
\]

would give us information about the intersection of these surfaces. We find \( z = \pm \sqrt{\frac{3}{2}} \). So the cone and the sphere intersect along the circles contained in the planes \( z = \pm \sqrt{\frac{3}{2}} \). It follows now that \( \phi \) within the upper half of the region \( W \) is goes from 0 to \( \arccos\left(\frac{\sqrt{3}/2}{\sqrt{3}/2}\right) = \frac{\pi}{4} \).

4. The Jacobian determinant, when changing from Cartesian to spherical coordinates, is equal to

\[
\left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = \rho^2 \sin \phi.
\]

5. We compute

\[
\int \int \int_{\text{upper half of } W} 1 \, dV = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{\rho(\sqrt{\frac{3}{2}})} r^2 \sin \phi \, d\rho d\phi d\theta = \int_0^{\frac{\pi}{4}} \int_0^{\rho(\sqrt{\frac{3}{2}})} \frac{\rho^3}{3} \sin \phi \bigg|_{\rho=0}^{\rho=\sqrt{\frac{3}{2}}} d\phi d\theta
\]

\[
= \int_0^{\frac{\pi}{4}} \int_0^{\rho(\sqrt{\frac{3}{2}})} \frac{\sqrt{3}}{2\sqrt{2}} \sin \phi \, d\phi d\theta = \frac{1}{2} \sqrt{\frac{3}{2}}(2 - \sqrt{2})\pi.
\]

Thus the total volume is \( \sqrt{\frac{3}{2}}(2 - \sqrt{2})\pi \) units.
6. Let $D$ be the region in the first quadrant of the $xy$-plane bounded by the curves $xy = 1$, $xy = 2$, $y = x$, $y = 4x$. Find the area of $D$.

Hint: Make an appropriate change of variables in the area integral.

**Solution.** This is how the region $D$ looks like:

![Diagram of the region D](image)

The area of $D$ is given by the double integral $\int \int_D 1 \, dA$. Within this region the quantity $xy$ varies from 1 to 2 and $\frac{y}{x}$ goes from 1 to 4. It suggests to consider the substitution $u = xy$, $v = \frac{y}{x}$ in the area integral.

Step 1. We let $u = xy$, $v = \frac{y}{x}$.

Step 2. As we already noticed, in terms of $u$ and $v$ the region $D$ is determined by the inequalities

$$1 \leq u \leq 2, \quad 1 \leq v \leq 4.$$ 

Step 3. Now we need to compute the Jacobian determinant $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|$. To do that, one can either express $x$ and $y$ as functions of $u$ and $v$ and compute the determinant directly or, alternatively, one can use the fact that

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \frac{\partial(u,v)}{\partial(x,y)} \right|^{-1}.$$ 

Let us consider both approaches.

i. From $u = xy$, $v = \frac{y}{x}$ we find $uv = y^2$, $\frac{u}{v} = x^2$. Since both $x$ and $y$ are assumed positive, then $y(u,v) = \sqrt{uv}$, $x(u,v) = \sqrt{\frac{u}{v}}$. We compute

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{\sqrt{uv}} & -\frac{1}{2} \sqrt{\frac{u}{v}} \\ \frac{1}{2} \sqrt{\frac{v}{u}} & \frac{1}{2} \sqrt{\frac{v}{u}} \end{array} \right| = \frac{1}{2v}.$$ 

ii. 

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \left| \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right| = \left| \begin{array}{cc} y & x \\ -\frac{x}{y} & \frac{1}{x} \end{array} \right| = \frac{2y}{x},$$

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left( \frac{\partial(u,v)}{\partial(x,y)} \right)^{-1} = \left( \frac{2y}{x} \right)^{-1} = \frac{1}{2v}.$$ 

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Step 4. We compute

\[
\int \int \int D 1 \, dA = \int_1^2 \int_1^4 1 \cdot \frac{1}{2v} \, dv \, du = \int_1^2 \frac{1}{2} \cdot \ln v \bigg|_1^4 \, du = \ln 2.
\]