Worksheet 12

How to evaluate a surface integral $\int \int_S f(x, y, z) dS$:

Step 1. Find a parametrization $T : D \rightarrow \mathbb{R}^3$ of the surface $S$. So $T(u, v) = \left( x(u, v), y(u, v), z(u, v) \right)$.

Step 2. Compute $T_u(u, v) = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$, $T_v(u, v) = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$ and $\|T_u \times T_v\|$. 

Step 3. Plug $x(u, v), y(u, v), z(u, v)$ for $x, y$ and $z$ respectively into $f(x, y, z)$.

Step 4. Evaluate the double integral $\int \int_D f \left( x(u, v), y(u, v), z(u, v) \right) \cdot \|T_u \times T_v\| \, du \, dv$.

1. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 100$ that lies between the planes $z = -8$ and $z = 6$.

   Hint: Recall that the area of $S$ is equal to the surface integral $\int \int_S 1 \, dS$.

2. A shell has the shape of the conical surface $x^2 + y^2 = z^2$, $0 \leq z \leq 1$. The density at point $(x, y, z)$ of this cone is equal to $\rho(x, y, z) = \sqrt{x^2 + y^2}$. Find the total mass of the shell.

   Hint: You can use cylindrical coordinates to parametrize the cone.
3. Evaluate the integral \( \iiint_S (x + y + z) \, dS \), where \( S \) is the part of the plane \( x + y + z = 1 \) for \( x, y, z \geq 0 \).

4. Evaluate the integral \( \iiint_S (xy + yz + zx) \, dS \) where \( S \) is the part of the cone \( z = \sqrt{x^2 + y^2} \) cut off by the cylinder \( x^2 + y^2 = 2z \).

5. The surface \( S \) has a parametrization \( \mathbf{T}(u, v) = (u \cos v, u \sin v, v), 0 \leq v \leq 2\pi, 0 \leq u \leq 1 \). Sketch it and evaluate the integral \( \iint_S z \, dS \).