**Worksheet 13\frac{1}{2}**

**Stokes’ Theorem**

Let $S$ be an oriented surface parametrized by a one-to-one function $\Phi : D \rightarrow \mathbb{R}^3$ and $\mathbf{F}$ be a vector field with continuously differentiable components. Denote by $\partial S$ the oriented (with respect to the right-hand rule) boundary of $S$. The Stokes’ theorem states that the surface integral $\int_S (\nabla \times \mathbf{F}) \, d\mathbf{S}$ is equal to the line integral $\int_{\partial S} \mathbf{F} \, d\mathbf{s}$.

When to use Stokes’ Theorem:

- Suppose, you need to compute the line integral of some vector field $\mathbf{F}$ over a **closed** curve $C$. On one hand, you may try to find a parametrization $\mathbf{c}$ of this curve and then set up and evaluate the line integral as $\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \, dt$.

- On the other hand, sometimes it may be easier to use Stokes’ theorem:
  1) find a surface $S$ which has $C$ as its boundary. There is a plenty of such surfaces. Choose the “most natural” one;
  2) set up and evaluate the surface integral $\int_S (\nabla \times \mathbf{F}) \, d\mathbf{S}$. At this step you will need to find a parametrization of $S$. That’s why it is important to choose a “nice” $S$. The value of this surface integral is exactly what we need. Just make sure that the orientations of $S$ and $C$ are chosen properly. Otherwise, you will get the answer with the wrong sign.

- Suppose, you need to compute the surface integral of the **curl** of some vector field $\mathbf{F}$ over a surface $S$. Parametrizing $S$ and evaluating the surface integral as a double integral may be tedious. Instead, you can try to employ Stokes’ theorem:
  1) determine, what curve is the boundary of the surface $S$. This curve is commonly denoted by $\partial S$. Take it with the proper orientation;
  2) set up and evaluate the line integral $\int_{\partial S} \mathbf{F} \, d\mathbf{s}$. You may need to find a parametrization of $\partial S$ here. The Stokes’ theorem guarantees that the value of this line integral is equal to the surface integral of curl $\mathbf{F}$ over $S$. Notice, that there are surfaces which have no boundary curves (for example, a sphere). In such a case, the surface integral $\int_S (\nabla \times \mathbf{F}) \, d\mathbf{S}$ is equal to zero.
Exercises

1. Using Stokes’ Theorem, evaluate the line integral \( \int_L \mathbf{F} \, ds \).
   
a) \( \mathbf{F}(x, y, z) = y^2 \mathbf{i} + z^2 \mathbf{j} + x^2 \mathbf{k} \); \( L \) is the triangle with the vertices \((a, 0, 0), (0, a, 0), (0, 0, a)\) positively oriented with respect to the vector \((0, 1, 0)\).
   
b) \( \mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k} \); \( L \) is the curve of intersection of the unit sphere centered at the origin and the plane \( x + y + z = 0 \) positively oriented with respect to the vector \((0, 0, 1)\).
   
c) \( \mathbf{F}(x, y, z) = x^2 y^3 \mathbf{i} + \mathbf{j} + z \mathbf{k} \); \( L \) is the circle \( x^2 + y^2 = R^2 \), \( z = 0 \) going counterclockwise.
   
d) \( \mathbf{F}(x, y, z) = x \mathbf{i} + (x + y) \mathbf{j} + (x + y + z) \mathbf{k} \); \( L \) is the curve parametrized by \( x = a \cos t \), \( y = a \sin t \), \( z = a(\sin t + \cos t) \), \( 0 \leq t \leq 2\pi \).
   
   Answer: 
   a) \(-a^3\); b) \(-\sqrt{3} \pi\); c) \(-\frac{\pi R^6}{8}\); d) \(-\pi a^2\).

2. Evaluate in two ways (with and without using Stokes’ theorem) the surface integral \( \int_S (\nabla \times \mathbf{F}) \, d\mathbf{S} \).
   
a) \( \mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k} \); \( S \) is the upper hemisphere \( x^2 + y^2 + z^2 = R^2 \), \( z \geq 0 \).
   
b) \( \mathbf{F}(x, y, z) = z(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \); \( S \) is the cylindrical surface \( x^2 + y^2 = 1 \), \(|z| \leq 1 \) oriented with outward pointing normal vectors.
   
c) \( \mathbf{F}(x, y, z) = z \mathbf{i} + x^2 \mathbf{j} + y \mathbf{k} \); \( S \) is the conical surface \( y^2 + z^2 = x^2 \), \( 0 \leq x \leq 1 \).

3. Let \( \mathbf{F} \) be the vector field defined by \( \mathbf{F}(x, y, z) = x(\mathbf{i} + \mathbf{j}) \) and \( C \) be the curve of intersection of the surface \( S \) given by the equation \( x^{10} + y^{10} + z^{10} = 5 \) and the cylinder \( x^2 + y^2 = 1 \), \( z \geq 0 \). Evaluate \( \int_C \mathbf{F} \, ds \).
   
   Hint: Apply Stokes’ Theorem to the portion of the given cylinder lying between the plane \( z = 0 \) and the surface \( S \).

   Answer: 