Let $W$ be a body enclosed by a surface $S$ with outward pointing unit normal $n$ and $F$ be a continuously differentiable vector field. **Gauss' divergence theorem** states that the surface integral $\int_S F \, dS$ is equal to the volume integral $\int \int \int \text{div} \, F \, dV$.

When to use Gauss’ theorem:

- Suppose, you want to evaluate the integral of a vector field $F$ over a **closed** surface $S$. By Gauss’ theorem, instead of parametrizing $S$ and setting up the surface integral as the double integral, you can determine what solid $W$ is enclosed by $S$, and then compute the triple integral of the divergence of $F$ over $W$.

- Suppose, you want to evaluate the integral of $\text{div} \, F$ over a solid region $W$. Instead of setting up the triple integral, you can determine what surface forms the boundary of $W$, and then compute the surface integral $\int \int_{\partial W} F \, dS$.

1. a) Find the flux of the vector field $\mathbf{F}(x, y, z) = (\mathbf{i} + \mathbf{j} + \mathbf{k})z^3$ through the unit sphere centered at the origin.

\[ \int \int_{S} F \, dS = 4\pi \]

b) Find the flux of the vector field $\mathbf{F}(x, y, z) = x^3 \mathbf{i} + y^3 \mathbf{j}$ through the entire surface of the solid cone $x^2 + y^2 \leq z^2$, $0 \leq z \leq 1$.

\[ \int \int_{S} F \, dS = \frac{3\pi}{10} \]

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$^1$Johann Carl Friedrich Gauss (1777-1855) was a German mathematician and scientist, who contributed significantly to many fields. Gauss is considered by many to be perhaps the greatest mathematician of all time.
2. Let $S$ be the surface enclosing some region $W$. It is known that the flux of the vector field $\mathbf{F}(x, y, z) = (y, x, z \sin^2 z)$ through $S$ is equal to 100 units per second and the flux of the vector field $\mathbf{G}(x, y, z) = (10, 10, z \cos^2 z)$ through the same surface is 200 units per second. What is the volume of $W$?

**Answer:** 300

3. a) Let $W$ be a solid region. Show that the flux of the vector field $\mathbf{r}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$ through the boundary surface of $W$ is equal to three times the volume of $W$.

b) A candy of volume $2\text{in}^3$ is small enough to fit into a sphere of radius 1in. Prove that it cannot be wrapped entirely in a 2-by-2in sheet of paper.

*Hint:* Use the result of part a). You may need to recall that $(\mathbf{a} \cdot \mathbf{b}) \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|$ for any vectors $\mathbf{a}$ and $\mathbf{b}$ (Schwarz’s inequality).

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\[2\text{Karl Hermann Amandus Schwarz (1843-1921) was a German mathematician.}\]