Worksheet 15\textsuperscript{3/4}

Mock Final

1. Find the angle between the plane \(x + y + z = 0\) and the line \(l(t) = (1 + 2t, -1 + 3t, 4 - t)\).

   \textit{Key words:} directing vector, normal vector, dot product.
   \textit{Answer:} \(\frac{\pi}{2} - \cos^{-1}\left(\frac{4}{\sqrt{42}}\right)\)

2. Parametrize the surface \(x^2 + z^2 + (y - 1)^2 = y^2, 2 \leq y \leq 3\).

   \textit{Key words:} quadratic surface, cross section, cylindrical coordinates.
   \textit{Answer:} \(T(r, \theta) = (r \cos \theta, \frac{r^2 + 1}{2}, r \sin \theta), \sqrt{3} \leq r \leq \sqrt{5}\)
   or \(T(\theta, y) = (\sqrt{2y - 1} \cos \theta, y, \sqrt{2y - 1} \sin \theta), 2 \leq y \leq 3\). Take \(0 \leq \theta \leq 2\pi\).

3. Find and classify the extrema points of the function \(z(x, y) = x^3 + 8y^3 - 6xy + 1\).

   \textit{Key words:} partial derivatives, critical point, Hessian matrix.
   \textit{Answer:} \(z\) has a minimum at \(\left(1, \frac{1}{2}\right)\) (there is also a saddle point at \((0, 0)\), but saddle points are not extreme points).

4. Find the 2-nd order Taylor polynomial for the function \(f(x, y) = \sqrt{x + y}\) centered at \((2, 2)\).

   \textit{Key words:} Taylor polynomial, partial derivatives.
   \textit{Answer:} \(P_2(h_1, h_2) = 2 + \frac{h_1}{4} + \frac{h_2}{4} + \frac{1}{2} \left(-\frac{h_1^2}{32} - \frac{h_1 h_2}{16} - \frac{h_2^2}{32}\right)\)

5. Find the plane tangent to the surface \(x^2 + 4y^2 + z^2 = 36\) and parallel to the plane \(x + y - z = 0\).

   \textit{Key words:} level surface, gradient, normal vector, equation of a plane.
   \textit{Answer:} there are two such planes: \(x + y - z - 1 = 0\) and \(x + y - z + 9 = 0\).

6. Compute the flux of the vector field \(\mathbf{F}(x, y, z) = (5x + y, 0, z)\) through the entire surface of the solid cone \(x^2 + y^2 \leq z^2, 0 \leq z \leq 4\).

   \textit{Key words:} surface integral, closed surface, Gauss’ divergence theorem, triple integral, cylindrical coordinates.
   \textit{Answer:} \(128\pi\).

7. Is there a function \(f\) such that \(\nabla f = \mathbf{F}\)? If \(f\) exists, find it.
   a) \(\mathbf{F}(x, y, z) = (ye^{xy}, xe^{xy} - z \sin(yz), -y \sin(yz))\); b) \(\mathbf{F}(x, y, z) = (y^2, -x^2, z^2)\).

   \textit{Key words:} partial derivatives, gradient, integration, conservative (gradient) field.
   \textit{Answer:} a) \(f(x, y, z) = e^{xy} + \cos(yz)\); b) no such \(f\) exists.
8. Evaluate the integral \( \int_C y \, dx + xz \, dy + yz \, dx \), where \( C \) has a parametrization \( c(t) = (e^{\cos t} + 5, e^{\sin 4t}, t^5 + 1) \), \( 0 \leq t \leq \pi \).

*Key words:* line integral, conservative field.
*Answer:* \( 5\pi^5 + \pi^5 \frac{1}{e} + \frac{1}{e} - e \).

9. Find the volume of the solid bounded by the surfaces \( x^2 + y^2 - z^2 = 0 \) and \( x^2 + y^2 + z^2 = 100 \).

*Key words:* quadric surfaces, sphere, cone, volume of a solid, triple integral, spherical coordinates.
*Answer:* \( 2 \cdot \frac{1000}{3} \pi (2 - \sqrt{2}) \) (the factor 2 appears, because the first surface is a double cone).

10. Evaluate the integral \( \iint_D y^2 \, dx \, dy \), where \( D \) is bounded by the curves \( x = y^2 \) and \( x - y = 2 \).

*Key words:* double integral, iterated integral, domain of integration.
*Answer:* \( \frac{63}{20} \).

11. Make the change of variables to evaluate the integral \( \iint_D x \, dy \, dx \), where \( D \) is the parallelogram with the vertices \((2, 1), (3, 0), (5, 1), (4, 2)\).

*Key words:* double integral, Jacobian.
*Answer:* 3.

12. The temperature in the room at point \((x, y, z)\) is given by \( T(x, y, z) = 10e^{-(x^2+y^2+z^2)} + 20z \). A fly is flying up along the path \( c(t) = \left( t \cos \pi t, t \sin \pi t, t^2 \right) \). How is the temperature changing for the fly at point \((-3, 0, 9)\)?

*Key words:* derivatives, rate of change, the chain rule.
*Answer:* increasing at the rate \( 120 - \frac{1140}{\cos \pi} \) degrees per second.

13. Evaluate the integral \( \iint_S z \, dS \), where \( S \) is the hemisphere \( x^2 + y^2 + z^2 = 10, \ y \geq 0 \).

*Key words:* surface integral of a scalar function, parametrization, spherical coordinates.
*Answer:* 0.

14. Evaluate the line integral \( \int_L (x+z) \, dx + (x-y) \, dy + x \, dz \), where \( L \) is the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), \( z = c \).

*Key words:* line integral, closed curve, Stokes’ theorem, curl.
*Answer:* \( \pm \pi ab \).

15. Evaluate the surface integral \( \iint_S \mathbf{F} \, d\mathbf{S} \), where \( S \) is the surface \( z = x^2 + y^2, \ 0 \leq z \leq 1 \) and \( \mathbf{F}(x, y, z) = (0, 0, z) \)

*Key words:* surface integral of a vector field, parametrization, cylindrical coordinates.
*Answer:* \( \pm \frac{\pi}{2} \).