Aus dem Paradies, das Cantor uns geschaffen, soll
uns niemand vertreiben können.  
David Hilbert, Über das Unendliche

Definition 1. A set is a collection of objects (called elements or members). We write
\( a \in A \) to mean that object \( a \) is an element of set \( A \), and we write \( a \notin A \) otherwise.

Examples.
(a) The set of students in class.
(b) The set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots \} \).
(c) The set of integers \( \mathbb{Z} = \{ \ldots, -1, 0, 1, 2, \ldots \} \).
(d) The set of real numbers \( \mathbb{R} \).
(e) A circle on the \( xy \)-plane \( C = \{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\} \).
(f) The set with no elements \( \emptyset \) (the empty set).

Definition 2. Let \( A \) and \( B \) be sets. If every element of \( A \) is an element of \( B \), we say that \( A \) is a subset of \( B \) and denote this by writing \( A \subseteq B \). Sets \( A \) and \( B \) are said to be equal (and we write \( A = B \)) if they consist of the same elements. Equivalently, \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \).

Basic operations with sets.
The union of sets \( A \) and \( B \) is the set \( A \cup B = \{x : x \in A \text{ or } x \in B\} \).
The intersection of sets \( A \) and \( B \) is the set \( A \cap B = \{x : x \in A \text{ and } x \in B\} \).
The complement of \( B \) in \( A \) is the set \( A \setminus B = \{x : x \in A \text{ and } x \notin B\} \).

1. Let \( A = \{1, 3, 7, 137\}, B = \{3, 7, 23\}, C = \{0, 1, 23\}, D = \{0, 7, 23, 2012\} \). Find
   (a) \((A \cap B) \cup D\)
   (b) \(A \cap (B \cup D)\)
   (c) \((A \cup B) \cap (C \cup D)\)
   (d) \((A \cup B) \setminus (C \cup D)\)
   (e) \(A \setminus (B \setminus (C \setminus D))\)

2. Let \( A = \{k \in \mathbb{Z} : k \text{ is divisible by 2}\}, B = \{k \in \mathbb{Z} : k \text{ is divisible by 3}\} \). Find
   (a) \(A \cap B\)
   (b) \(\mathbb{Z} \setminus A\)
   (c) \(\mathbb{Z} \setminus (A \cup B)\)

3. For each \( i \in \mathbb{N} \), let \( A_i = \left[-\frac{1}{n}, \frac{1}{n}\right] \). Find
   (a) \(\bigcap_{i=1}^{20} A_i\)
   (b) \(\bigcup_{i=20}^{\infty} A_i\)
   (c) \(\bigcap_{i=1}^{\infty} A_i\)

1"No one should be able to drive us out of the paradise that Cantor created for us." - David Hilbert on

2It is very naive and has some serious flaws (see section 9 of the textbook, if you are interested), but it
should be OK for our needs.
4. For each of the following identities, prove or give a counterexample.

Advice. Drawing Venn diagrams might be helpful (although it would not count as a rigorous proof).

a) \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)
b) \( A \setminus (B \setminus C) = (A \setminus B) \cap (A \setminus C) \)
c) \( (A \setminus B) \cup B = A \)
d) \( A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C) \).

5. Let \( A \) and \( B \) be sets. Is it possible to find the intersection \( A \cap B \) using only the operations of taking the union and the complement (that is, \( \cup \) and \( \setminus \)?)

Definition 3. Let \( A \) and \( B \) be sets. The Cartesian product, written \( A \times B \), is the set of all ordered pairs \( (a, b) \), where \( a \in A, \ b \in B \). A relation between sets \( A \) and \( B \) is a subset \( R \subseteq A \times B \).

6. Let \( A, B, C, D \) be as in exercise 1. How many elements is in the set \( A \times B \)? What about \( (B \times C) \cap (A \times C) \)

7. In a class of thirty people, each student solved three problems from a worksheet and each problem on this worksheet was solved by exactly ten students. How many problems were on the worksheet?

Hint. Let \( S \) be the set of students and \( P \) be the set of problems on the worksheet. Consider a relation \( R \) between \( S \) and \( P \) defined by \( xRy \iff \text{student } x \text{ solved problem } y \). How many elements is in \( R \)?

8. Prove or give a counterexample.

a) \( (A \setminus B) \times C = (A \times C) \setminus (B \times C) \).
b) \( (A \cap B) \times C = (A \cap C) \times (B \cap C) \)
c) \( (A \cap B) \times C = (A \times C) \cap (B \times C) \)

\(^3\)Named after the eminent French philosopher and mathematician René Descartes (1596-1650).