Definition 1. A relation $R \subseteq S \times S$ is called an equivalence relation if it is
- reflexive: for any $x \in S$, $xRx$;
- symmetric: if $x, y \in S$ are such that $xRy$, then $yRx$;
- transitive: if $x, y, z \in S$ are such that $xRy$ and $yRz$, then $xRz$.

Definition 2. Let $R$ be an equivalence relation on a set $S$ and $x \in S$. Then the set of all elements of $S$, which are equivalent to $x$ with respect to the relation $R$, is called the equivalence class of $x$ and denoted by $E_x$. More formally, $E_x = \{ y \in S | xRy \}$. An element $y \in E_x$ is called a representative of the class $E_x$.

1. Let $P$ be the set of all living people. Define a relation $B$ on $P$ by the rule $xB y \Leftrightarrow x$ and $y$ have the same birthday. Verify that $B$ is an equivalence relation. How many equivalence classes for the relation $B$ are there? How many elements (approximately) is in the equivalence class $E_{\text{Bob Dylan}}$?

2. Define a relation on $\mathbb{N} \times \mathbb{N}$ by the rule $(a, b)R(c, d) \Leftrightarrow a^b = c^d$.
   a) Prove that $R$ is an equivalence relation.
   b) What are the elements of the equivalence class $E_{(9,2)}$?
   c) Find an equivalence class with exactly one element. Find an equivalence class with exactly three elements. Is there an equivalence class, which has infinitely many elements?

Definition 3. Let $A$ and $B$ be sets. A function (or a mapping) $f$ from $A$ to $B$ (commonly denoted by $f : A \rightarrow B$) is a relation $f \subseteq A \times B$ such that
- for any $a \in A$, there is $b \in B$ such that $(a, b) \in f$;
- if $(a, b), (a, c) \in f$, then $b = c$.

One can think of $f$ as of a rule, which associates an element of set $B$ to each element of set $A$. In particular, if $(a, b) \in f$ then we say that the value of $f$ at $a$ is equal to $b$ and write $f(a) = b$. The set $A$ is called the domain of a function $f : A \rightarrow B$ and $B$ is called the codomain of $f$. The range of $f$ is the set of all values that $f$ takes.

Example. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by the rule $f(x) = \sin(x)$ for $x \in \mathbb{R}$. Then both domain and codomain of $f$ is the set of real numbers $\mathbb{R}$, but the range of $f$ is just the closed interval $[-1; 1].$

Definition 4. We give names to some important classes of functions
- A function $f : A \rightarrow B$ is called surjective (or onto) if the range of $f$ is equal to the entire set $B$.
- A function $f : A \rightarrow B$ is called injective (or one-to-one) if it maps distinct elements of $A$ to distinct elements of $B$. More formally, it means that an equality $f(a) = f(a')$ implies that $a = a'$.
- A function $f : A \rightarrow B$ is called bijective if it is both surjective and injective.

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1 Alternative notations: $[x]_{\mathbb{R}}, [x], \bar{x}$.
2 From French sur - 'on', 'onto', 'over', 'above'. Compare with surveillance or surplus.
3. For each of the following functions, determine if it is surjective, injective, bijective or none of these.
   (a) \( f : \{ \text{the U.S. citizens}\} \to \mathbb{N} \) defined by \( f(x) = \text{SSN of } x \).
   (b) \( f : \{ \text{all people in the world}\} \to \{ \text{Jan.1, Jan.2, \ldots, Dec.31}\} \) defined by \( f(x) = \text{the birthday of } x \).
   (c) \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^3 - 3 \)
   (d) \( f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \) defined by \( f(n, m) = n^2 + m^2 \)
   (e) \( f : \mathbb{Z} \to \mathbb{N} \times \mathbb{N} \) defined by \( f(k) = (k^2, k + 3) \)

4. Find a function \( f : \mathbb{N} \to \mathbb{N} \), which has the desired properties.
   (a) surjective, but not injective;
   (b) injective, but not surjective;

5. For each of the following statements, prove it or give a counterexample.
   a) \( f(A \cup B) = f(A) \cup f(B) \)
   b) \( f(A \cap B) = f(A) \cap f(B) \)
   c) if \( A \subset B \), then \( f(A) \subset f(B) \)
   d) if \( f(A) \subset f(B) \), then \( A \subset B \)
   e) \( f(A \setminus C) \subset f(A) \setminus f(C) \)
   f) \( f(A \setminus C) \subset f(A \setminus C) \)