Men usually infer from this mirror that the Library is not infinite [...]; I prefer to dream that its polished surfaces represent and promise the infinite.

Jorge Luis Borges, *The Library of Babel*

**Definition 1.** Sets $A$ and $B$ are said to have the same *cardinality* if there exists a bijection between them. In such a case, we write $|A| = |B|$. If there exists an injection $A \to B$, then we say that the cardinality of $A$ is less than or equal to the cardinality of $B$ and denote it by $|A| \leq |B|$.

**Example.**

a) Sets $\{1, 2, 3\}$ and $\{\star, \▲, \♠\}$ have the same cardinality.
b) Sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality ("aleph zero"). It is also true that $|\mathbb{Q}| = |\mathbb{N}|$ (see example 8.11(c)). In general, sets having the same cardinality as $\mathbb{N}$ are called *denumerable.*
c) The set of real numbers $\mathbb{R}$ is not denumerable; it has cardinality strictly greater than $|\mathbb{N}|$ (Theorem 8.12).

**Theorem** (E.Schröder, F.Bernstein). If there exist injective functions $A \to B$ and $B \to A$, then there exists a bijective function from $A$ to $B$. In terms of cardinalities, the theorem can be stated as follows: if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Two basic ways to prove that sets $A$ and $B$ have the same cardinality:

- Construct an explicit bijection $A \to B$ (or $B \to A$).
- Construct injections $A \to B$ and $B \to A$. If you succeed with this, then the Schröder-Bernstein theorem will imply that $|A| = |B|$.

1. Show that the following pairs of sets $A$ and $B$ have the same cardinality by finding a bijection between them.

a) $A$ is the set of all odd integers and $B$ is the set of all even integers.
b) $A = [0, 1], B = [2, 4]$
c) $A = [0, 1), B = (100, 200]$ 
d) $A = (0, 1), B = (0, \infty)$.
e) $A = \mathbb{R}, B = (1, \infty)$.
f) $A = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), B = \mathbb{R}$.
g) $A$ is the boundary of the square $[-1, 1] \times [-1, 1], B$ is a circle of radius 1 centered at the origin.
2. Use the Schöder-Bernstein theorem to prove that sets $A$ and $B$ have the same cardinality.

   a) $A = [0, 1] \cup [2, 3], \ B = [0, 1]$

   b) $A = \mathbb{R}, \ B = \mathbb{R} \setminus \{0\}$.

   c) $A = \mathbb{Z}, \ B = \mathbb{Z} \times \mathbb{Z}$.

   d) $A = \blacksquare, \ B = \blacklozenge$.

3. Let $A$, $B$, $C$ and $D$ be sets such that $|A| = |B|$ and $|C| = |D|$. Prove that $|A \times C| = |B \times D|$.

4. Prove that if $A$ is denumerable and $B$ is countable, then $A \times B$ is also denumerable.

5. Prove that if $A$ is infinite and $B$ is countable, then $A \cup B$ is infinite.