Principle of Mathematical Induction.
Let $P(n)$ be a proposition depending on a natural number $n \in \mathbb{N}$. If it is the case that
- $P(1)$ is true;
- for any $k \in \mathbb{N}$, the assumption that $P(k)$ is true implies that $P(k+1)$ is true,
then $P(n)$ is true for any $n \in \mathbb{N}$.

The statement $P(1)$ is referred to as the basis of induction and the implication $P(k) \Rightarrow P(k+1)$ is called the induction step.

1. (warm-up) Prove that for any natural $n$, the following identity holds:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

1. a) Prove that for any natural $n$, the number $15^n + 6$ is divisible by 7.

b) Prove that for any natural $n$, the number $n^3 + 5n$ is divisible by 6.

2. Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{2012^2} < \frac{2011}{2012}$. 


Jorge Luis Borges, The Library of Babel
3. Recall that a polygon $P$ is called \textit{convex} if any diagonal of $P$ lies entirely inside $P$. Prove that the sum of the internal angles of a convex polygon with $n$ vertices is equal to $180(n - 2)$ degrees.

4. Consider a $2^n$-by-$2^n$ chessboard with a corner square removed. Prove that it can be tiled without gaps and overlaps by L-shaped figures consisting of 3 squares each (the figures can be rotated).

5. Prove that for any $n \geq 6$, a square can be cut into $n$ squares.

6. There are $n$ lines drawn in the plane in such a way that no two are parallel and no three intersect in a common point (see the picture for the case $n = 3$). Prove that these $n$ lines divide the plane into $T_n = \frac{n(n+1)}{2} + 1$ parts.