Consider the following variation of Newton's method for obtaining an approximation to \( \sqrt{2} \):

\[(1.14) \quad \text{Take } x^0 \text{ to be a positive number.} \]
\[(1.15) \quad \text{Set } x^{k+1} = x^k + \beta \frac{A}{x^k} \quad \text{for } k \geq 0.\]

(2pt) **(1)** For what values of \( \alpha \) and \( \beta \) is the method consistent?

(2pt) **(2)** For what values of \( \alpha \) and \( \beta \) is the method both consistent and stable?

(2pt) **(3)** For what values of \( \alpha \) and \( \beta \) is the method convergent?

(2pt) **(4)** For what values of \( \alpha \) and \( \beta \) does the method converge faster? Provide numerical results backing your answer.

(3pt) **(5)** Represent all elements of \( \mathbb{R}(3, 2) \).

(4pt) **(6)** Given any machine, how would you find the parameters \( \alpha \) and \( \beta \) determining its set of numbers \( \mathbb{R}(s, t) \)? Provide numerical results supporting your idea.

Next, we want to consider our algorithm (1.11). For each \( \varepsilon \) and \( A \), we are going to fill the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x^0 )</th>
<th>( x^1 )</th>
<th>( \frac{(x^1 - x^0)}{(x^1 - x^0)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>...</td>
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</tr>
<tr>
<td>2</td>
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<td>...</td>
</tr>
<tr>
<td>N</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
(3ph) ⑦ Take $A = 5$ and $e = 10^{-2}$. Obtain the history of convergence table. Do the results match the theory? Now take $e = 10^{-4}$ and then $10^{-2}$. Explain your results.

(3ph) ⑧ Repeat the above exercise with $A = 5 \cdot 10^2$. 
We want to study the following variation of Newton's method for obtaining an approximation to \( f'(x) \):

\[
\begin{align*}
(1.1a) & \quad \text{Take } x^0 \text{ to be a positive number.} \\
(1.1b) & \quad \text{Set } x^{k+1} = x^k - \frac{1}{\beta} \frac{A}{x^k} \quad \text{for } k > 0,
\end{align*}
\]

for some constants \( \alpha \) and \( \beta \). We want to know for what values of these two constants the method (1.1) is consistent, stable, and convergent. We also want to know for what values it converges faster.

We say that the method is consistent if the exact solution \( \{ x^k = f'(x) \} \) satisfies (1.1). It is not difficult to see that (1.1a) is satisfied and that (1.1b) is satisfied if and only if \( \alpha + \beta = 1 \).

Now let us consider the more difficult issue of the stability of the method with respect to its initial guess. By (1.1b), we have

\[
\begin{align*}
X_{1}^{k} & = x_{1}^{k} - \frac{1}{\beta} \frac{A}{x_{1}^{k}} \quad \text{for } k > 0, \\
X_{2}^{k} & = x_{2}^{k} - \frac{1}{\beta} \frac{A}{x_{2}^{k}} \quad \text{for } k > 0,
\end{align*}
\]

and so, for \( S := X_{1}^{k} - X_{2}^{k} \),

\[
S^{k+1} = - \frac{A}{x_{2}^{k}} \left( \frac{X_{1}^{k}}{x_{2}^{k}} - \frac{X_{2}^{k}}{x_{2}^{k}} \right) = - \frac{A}{x_{2}^{k}} S^{k},
\]

for \( k > 0 \).
\[ S^k = \alpha S^k + \beta A \left( \frac{1}{x^k_1} - \frac{1}{x^k_2} \right) \]
\[ = \alpha S^k - \beta A \frac{S^k}{x^k_1 x^k_2} \]
\[ = \left( \alpha - \frac{\beta A}{x^k_1 x^k_2} \right) S^k \quad \text{for } k \geq 0. \]

To obtain a bound for the factor \((\alpha - \frac{\beta A}{x^k_1 x^k_2})\), we proceed as follows. First, note that \(\alpha\) and \(\beta\) have to be nonnegative parameters. If not, if we take in (1.10) the initial guess

\[ x^0 = \sqrt{-\frac{\beta A}{\alpha}}, \]

then \(x^0 = 0\) and \(x^2\) is not defined. (Note that we are assuming that the method is consistent, that is, that \(\alpha + \beta = 1\). So, if \(\alpha + \beta\) is negative, we immediately have that \(x \neq 0\) and \(\beta - \frac{\beta A}{x^2} < 0\).

This shows that we only need to consider the case \(\beta = 1 - \alpha\) and \(\alpha \in [0, 1]\). If \(\alpha = 0\), then \(\beta = 1\) and we have that (1.10) generates the sequence

\[ x^0, \quad x^1 = \frac{A}{x^0_1}, \quad x^2 = x^0, \quad x^3 = \frac{A}{x^0_2}, \ldots. \]

This means that we have

\[ \alpha - \frac{\beta A}{x^k_1 x^k_2} = \begin{cases} -\frac{A}{x^k_1 x^k_2} & \text{for } k \text{ even}, \\ -\frac{A}{x^k_1 x^k_2} & \text{for } k \text{ odd}, \end{cases} \]
\[ \delta_1 = -\frac{A}{x_1 x_2} \delta^0 \]
\[ \delta_2 = -\frac{x_1^2 x_2^2}{A} \delta_1 = \delta^0 \]
\[ \delta_3 = -\frac{A}{x_1^2 x_2} \delta_2 = -\frac{A}{x_1^2 x_2} \delta^0 \]
\[ \delta_4 = \delta^0. \]

and so on. This implies that

\[ |\delta^{k+1}| \leq \max \left\{ 1, \frac{A}{x_1 x_2} \right\} |\delta^0| \quad \text{for} \ k \geq 0. \]

We thus see that the method is stable for \( \alpha = 0 \) and \( \beta = 1 \).

Next, we consider the case \( \alpha \in (0, 1) \). In this case, we use the fact that, by (1.16),

\[ x_k^{k+1} = \alpha x_k^k + \beta \frac{A}{x_k^k} \]

\[ \geq \min_{x > 0} \left( \alpha x + \beta \frac{A}{x} \right) \]

\[ = 2 \sqrt{\alpha \beta A} \quad \text{for} \ k \geq 0. \]

This immediately implies that
\[ \left( \alpha - \frac{\beta A}{X_1 X_2} \right) \in \left[ \alpha - \frac{1}{4\alpha}, \alpha \right] \quad \text{for } k \geq 1. \]

Hence
\[ |s^{k+1}| \leq \max \left\{ \alpha, \left| \alpha - \frac{1}{4\alpha} \right| \right\} |s^k| \quad \text{for } k \geq 1, \]
\[ |s^1| \leq |\alpha - \frac{\beta A}{X_1 X_2}| |s^0|. \]

Set \( p = \max \left\{ \alpha, \left| \alpha - \frac{1}{4\alpha} \right| \right\} \). Then we have that
\[ |s^{k+1}| \leq p^k \left| \alpha - \frac{\beta A}{X_1 X_2} \right| |s^0| \quad \text{for } k \geq 0. \]

To obtain stability, it is enough to require that the parameter \( p \) be less than one. A little computation shows that this happens for \( \alpha \in \left( \frac{\sqrt{5} - 1}{2}, 1 \right) \).

Indeed, \( p = 1 \) if and only if \( \alpha = 1 \) or \( \left| \alpha - \frac{1}{4\alpha} \right| = 1 \). The latter case takes place if and only if
\[ \frac{1}{4\alpha} - \alpha = 1, \]
that is, if and only if
\[ 4\alpha^2 + 4\alpha - 1 = 0 \]

hence, if and only if \( \alpha = \frac{\sqrt{5} - 1}{2} \). Since \( \alpha > 0 \), we take \( \alpha = \frac{\sqrt{5} - 1}{2} \). The parameter \( p \) is not bigger than one if \( \alpha \) lies in \( \left( \frac{\sqrt{5} - 1}{2}, 1 \right) \) as can be readily verified.
the method is thus stable for $\alpha \in (\sqrt{2} - 1, 1]$. Note that $\frac{1}{2}(\sqrt{2} - 1) \approx 0.2$.

To obtain the stability of the method for $\alpha \in (0, \frac{\sqrt{2} - 1}{2})$ does not seem to be as simple as the previous case.

Let us now deal with the issue of convergence. We thus use the fact that the method is consistent. To replace $(\bar{x}^k)^{k=0}$ by the exact solution. Then dropping the subindex \( \bar{2} \) from $x^k_2$, we obtain from our previous results on stability:

$$\left| \sqrt{A} - x^{k+1} \right| \leq \max \left\{ 1, \frac{\sqrt{A}}{x^0} \right\} \left| \sqrt{A} - x^0 \right| \text{ for } k \geq 0$$

for $\alpha = 1$, and

$$\left| \sqrt{A} - x^{k+1} \right| \leq \left( \frac{\sqrt{A}}{x^0} \right)^k \left| \sqrt{A} - x^0 \right| \text{ for } k \geq 0$$

and $\alpha \in (0, 1]$. From these results, we can conclude that the method converges for $\alpha \in (\sqrt{2} - 1, 1)$. Note that the method does not converge for $\alpha = 0$ and for $\alpha = 1$! With the approach we have used, we cannot say anything about the convergence of the method for $\alpha \in (0, \frac{\sqrt{2} - 1}{2})$.

Finally, let us consider for what values of $\alpha$ and $\beta$ the method converges faster. Since we have
\[
\sqrt{A} - x = \left( x - \frac{\sqrt{A}}{x^k} \right) \left( \sqrt{A} - x^k \right)
\]

we can see that the error is reduced faster the smaller is the factor \((x - \sqrt{A}/x^k)\). If we assume that the sequence \(\{x^k\}_{k \geq 0}\) converges to \(\sqrt{A}\), such factors also converge to \((x - \sqrt{A})\). This means that the smaller \(|x - \sqrt{A}|\) the faster the convergence of the method should be. Clearly this happens when \(x = \sqrt{A}\). Since \(\beta = 1-x\), this implies that we expect the fastest convergence for \(\sqrt{A} = \beta = \frac{1}{2}\).