Consider the following model for European call options:

\[
\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf = 0 \quad \text{in} \, (0,S_M) \times (0,T),
\]

\[
\begin{align*}
E(S,T) &= \max\{S - K, 0\} \
\hat{E}(0,t) &= 0 \
\hat{E}(S, t) &= S_M - K e^{-r(T-t)}
\end{align*}
\quad \forall s \in (0, S_M), \\
\forall t \in (0, T), \\
\forall S \in (0, S_M).
\]

Take \( K = 1 \), \( S_M = 2 \), \( r = 0.1 \) and \( T = 0.2 \). Find the value of the option at time \( t = 0 \).

(8pts) Do this by transforming our problem to a problem involving a heat equation for the function \( F \) (as in page 3 of the notes on finite differences), solving it numerically with the Crank-Nicolson scheme, and then finding \( f \) from the values of \( F \).

(Note that, as indicated on page 2, we have to carry out the change of variable \( S = e^x \), which maps the interval \( (0, S_M) \) into \( (-\infty, \ln S_M) \). Since we cannot use finite differences on such an interval, we must replace it by an interval of the form \( (-x_0, \ln S_M) \). Accordingly, the boundary conditions at \( -x_0 \) would then be \( E(-x_0, t) = 0 \) \( \forall t \in (0,T) \). You would have to choose \( x_0 \) so that...
your approximation to \( f(\cdot, t=0) \) is sufficiently accurate. If \(|x_0|\) is too small, the approximation might not be good enough and if \(|x_0|\) is very big you might be wasting computational resources. Carry out several experiments to conclude with confidence that your choice of \( x_0 \) is sensible.

(16pt) 2. Devise a finite difference scheme for the model (E) for European call options without transforming it into a heat equation. Find an approximation to the value of the option at time \( t=0 \). Compare this approximation with the one obtained in the previous exercise.