Homework # 5  MSE 5012 (Due on May 1, 2008).

Consider the solution \( u \in C^1(0,1) \) of the following obstacle problem:

\[
\begin{align*}
(-u'' - 1) (u - f) &= 0 & \text{in} & \ (0,1), \\
(-u'' - 1) &> 0 & \text{in} & \ (0,1), \\
(u - f) &> 0 & \text{in} & \ (0,1), \\
u(0) &= u(1) = 0
\end{align*}
\]

where

\[
f(x) = 1 - 8 \left(x - \frac{1}{2}\right)^2.
\]

(4pt) (2) Find the exact solution of the obstacle problem.

(8pt) (2) Consider the following method for numerically solving the obstacle problem:

(a) Set \( u_i^{(0)} = 2 - 8 \left(i \, \Delta x - \frac{1}{2}\right)^2 \) \( i = 0, \ldots, I \).

(b) Given \( \{ u_i^{(k)} \}_{i=0}^I \), compute

\[
u_i^{(k+1)} = \max \left\{ f_i, u_i^{(k)} + \frac{\omega}{2} \left( \Delta x^2 + (u_{i+1}^{(k)} - 2u_i^{(k)} + u_{i-1}^{(k)}) \right) \right\}
\]

for \( i = 1, \ldots, I-1 \).

\[
u_0^{(k+1)} = u_0^{(k)} = 0. \text{ Here } f_i = f(i \, \Delta x) \text{ and } \Delta x = \frac{1}{I}.
\]

Show that the method converges to \( \{ u_i \}_{i=0}^I \), where

\[
\begin{align*}
\left( -\frac{1}{\Delta x^2} (u_{i+1}^{*} - 2u_i^{*} + u_{i-1}^{*}) - 1 \right) (u_i^{*} - f_i^{*}) &= 0 & \text{for} & \ i = 1, \ldots, I-1, \\
\left( -\frac{1}{\Delta x^2} (u_{i+1}^{*} - 2u_i^{*} + u_{i-1}^{*}) - 1 \right) &> 0 & \text{for} & \ i = 1, \ldots, I-1, \\
(u_i^{*} - f_i^{*}) &> 0 & \text{for} & \ i = 1, \ldots, I-1.
\end{align*}
\]
and

\[ u_0 = u_1 = 0, \]

provided \( \omega \in (0, 1) \).

(4.55) 3. For \( I = 40 \) and \( \omega = \frac{1}{2} \), plot several intermediate
functions \( \{ u^{(k)} \}_{k=0}^J \). Is the method converging?
Are your results matching with the theory? Why, or
why not?

(4.55) 4. Fill the table of convergence below

<table>
<thead>
<tr>
<th>I</th>
<th>( | e |_{L^1} )</th>
<th>N</th>
<th>( \chi )</th>
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<tbody>
<tr>
<td>4</td>
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<td>32</td>
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</tbody>
</table>

where \( \| e \|_{L^1} = \max_{0 \leq t \leq 1} \left| u(t, \frac{t}{I}) - u(t) \right| \), \( N \) is the
number of iterations needed to converge and
\( \chi \) is the estimated order of convergence. Try
\( \omega = \frac{1}{2} \), \( \omega = 1 \) and \( \omega = \frac{3}{2} \). How does the
variation in \( \omega \) affect \( N \) and \( \chi \)?