Sketch of the solution of mid-term #1

1. To find the solution, we separate variables to obtain

\[ \frac{dy}{y^{\frac{1}{2}}} = dt \]

Since \( \ln |y^{\frac{1}{2}}| = \frac{dy}{y^{\frac{1}{2}}} \), we get that

\[ \ln |y^{\frac{1}{2}}| = t + C, \]

and no, \( |y^{\frac{1}{2}}| = ke^t \). At \( t = 0 \), \( y = 2 \) and we get that \( \frac{1}{2} + \frac{3}{2} e^t \).

2. Since \( 1 + \cos y \geq 0 \) for any \( y \in \mathbb{R} \), and \( 1 + \cos y = 0 \) only for \( y = \pi + 2k\pi \) for all integers \( k \), the phase diagram is the following:

\[ \begin{align*}
&\quad \uparrow 5\pi \\
&\quad \uparrow 3\pi \\
&\quad \uparrow \pi \\
&\quad \uparrow 0 \\
&\quad \uparrow -\pi \\
&\quad \uparrow -3\pi \\
&\quad \quad \quad \text{the equilibrium points} \\
&\quad \quad \quad y = \pi + 2k\pi, \ k \in \mathbb{Z} \\
&\quad \quad \quad \text{are all nodes.}
\end{align*} \]

3. For the equation \( \frac{dy}{dt} = (y^{\frac{1}{2}})(y+1) \), we have that

\( y = -\frac{1}{2} \) is an equilibrium point. Also, we have that

\[ \frac{d}{dt} y = \begin{cases} 
> 0 & \text{if } y^{\frac{1}{2}} > 0 \text{ and } y + 1 > 0 \quad \text{(I)} \\
< 0 & \text{if } y^{\frac{1}{2}} < 0 \text{ and } y + 1 < 0 \quad \text{(II)} \\
\quad \quad \quad \quad \text{and } y + 1 > 0 \quad \text{(III)} \\
< 0 & \text{if } y^{\frac{1}{2}} < 0 \text{ and } y + 1 < 0 \quad \text{(IV)} \\
\quad \quad \quad \quad \text{and } y + 1 > 0
\end{cases} \]

Next, we sketch four solutions of this equation.
the solution passing through $\frac{1}{2}$ at $t=0$ has its
time derivative increasing with both time and "y".
So, at $t \rightarrow \infty$, $y(t)$ will also tend to $\infty$.

(4) Set $f_r(y) = r + 4y - y^3$. The equilibrium points are
the zeroes of $f_r(y)$ and the bifurcation points are
those equilibrium points for which $\frac{dy}{dy}f_r(y) = 0$. Hence
we have to solve the equations

$$r + 4y - y^3 = 0$$

$$4 - 3y^2 = 0$$

the second equation gives $y = \pm \frac{2}{\sqrt{3}}$ and the first
given $r = -4 \left( \pm \frac{2}{\sqrt{3}} \right) + \left( \pm \frac{2}{\sqrt{3}} \right)^3 = -\frac{3}{\sqrt{3}} + \frac{8}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} (\frac{2}{3}) = -\frac{16}{3\sqrt{3}}$