Math 4512  HW #4

1.

\[ y'' + \omega_0^2 y = 0 \]  where  \( \omega_0^2 = k/m \).

\[ k \Delta l = mg \Rightarrow k = mg/\Delta l \]

\[ = 9.8 \times 1/(49/320) \]

\[ = 64 \]

The general solution of \( y'' + \omega_0^2 y = 0 \) is

\[ y(t) = a \cos \omega_0 t + b \sin \omega_0 t \]

\[ = R \cos(\omega_0 t - \delta) \]  by Lemma 1

where  \( R = \sqrt{a^2 + b^2} \)  and  \( \delta = \tan^{-1} b/a \).

\[ y(0) = \frac{1}{4} \]  and  \( y'(0) = 0 \)  \( \Rightarrow a = \frac{1}{4}, b = 0 \)

Hence,  \( y(t) = \sqrt{(\frac{1}{4})^2 + 0} \cos(\sqrt{64}t - 0) \)

\[ = \frac{1}{4} \cos(8t) \]

Frequency  \( \omega_0 = 8 \)

Period  \( T_0 = 2\pi/\omega_0 = \pi/4 \)

Amplitude  \( R = \frac{1}{4} \)
Let $y(t)$ be the position of the mass away from the equilibrium position at $t$

$m = 1, \quad k = 1, \quad c = 2$

Hence,

$$y'' + 2y' + y = 0 \quad ; \quad y(0) = \frac{1}{4}, \quad y'(0) = -1$$

The characteristic equation is

$$r^2 + 2r + 1 = 0$$

$$\Rightarrow r_1 = r_2 = -1$$

The general solution of ODE is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

$$y(0) = \frac{1}{4}, \quad y'(0) = -1 \Rightarrow C_1 = \frac{1}{4}, \quad C_2 = -\frac{3}{4}$$

Hence,

$$y(t) = \frac{1}{4} e^{-t} - \frac{3}{4} t e^{-t} = \frac{1}{4} e^{-t} (1 - 3t)$$

At $t = \frac{1}{3}$, $y(\frac{1}{3}) = 0$ and $t = \frac{1}{3}$ is the only solution that satisfies $y(t) = 0$. So this means the mass will overshoot its equilibrium position once. On the other hand, $y(t) \rightarrow 0$ as $t \rightarrow \infty$, which means the mass will creep back to equilibrium as $t \rightarrow \infty$. 
5.

\[ y'' - 5y' + 4y = e^{2t}; \quad y(0) = 1, \ y'(0) = -1 \]

Solution:

Let \( Y(s) = \mathcal{L}\{y(t)\} \). Taking Laplace transform of both sides
of the differential equation gives

\[ s^2 Y(s) - s + 1 - 5[sY(s) - 1] + 4Y(s) = \frac{1}{s-2} \]

(Notice that \( \mathcal{L}\{y'(t)\} \) and \( \mathcal{L}\{y''(t)\} \) are computed by lemmas 3 and 4)

and implies that

\[ Y(s) = \frac{(s-6)(s-2) + 1}{(s-2)(s-1)(s-4)} \]

\[ = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-4} \]

\[ \Rightarrow A(s-2)(s-4) + B(s-1)(s-4) + C(s-1)(s-2) = (s-2)(s-6) + 1 \]

\( s = 2 \) \( \Rightarrow B = -\frac{1}{2} \), \( s = 4 \) \( \Rightarrow C = -\frac{1}{2} \), \( s = 1 \) \( \Rightarrow A = 2 \)

Thus, \( Y(s) = 2 \frac{1}{s-1} - \frac{1}{2} \frac{1}{s-2} - \frac{1}{2} \frac{1}{s-4} \)

Therefore, \( Y(s) = \mathcal{L}\{2e^{2t} - \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t}\} \)

so that

\[ y(t) = 2e^{2t} - \frac{1}{2}e^{2t} - \frac{1}{2}e^{4t} \]
There is also an alternative way to do this.

\[ y(t) = Ae^{2t} \]

is a very good guess of the particular solution.

Thus, \( y'(t) = 2Ae^{2t} \) and \( y''(t) = 4Ae^{2t} \)

\[ 4Ae^{2t} - 10Ae^{2t} + 4Ae^{2t} = e^{2t} \]

\( \Rightarrow A = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2} e^{2t} \)

To solve the corresponding homogeneous equation

\[ y'' - 5y' + 4y = 0 \]

we note that

\( r^2 - 5r + 4 = 0 \Rightarrow r = 1 \) and \( 4 \)

The general solution of \( y'' - 5y' + 4y = 0 \) is

\[ y_h = C_1 e^{t} + C_2 e^{4t} \]

\( \Rightarrow y(t) = y_p + y_h = -\frac{1}{2} e^{2t} + C_1 e^{t} + C_2 e^{4t} \)

\( y(0) = 1 \) and \( y'(0) = -1 \Rightarrow y(t) = 2e^{t} - \frac{1}{2} e^{4t} - \frac{1}{2} e^{2t} \)

\[ \text{Notice: I do not recommend this unless you have trouble with Laplace transform.} \]
21. 

\[ y'' - 2y' + y = te^t; \quad y(0) = 0, \quad y'(0) = 0 \]

The Laplace transform of \( e^t \) is \( 1/(s-1) \). Hence,

\[ \mathcal{L}\{te^t\} = - \frac{d}{ds} \frac{1}{s-1} = \frac{1}{(s-1)^2} \]

by property 1.

Then, we apply the same method with last problem

\[ s^2 Y(s) - 2sY(s) + Y(s) = \frac{1}{(s-1)^2} \]

\[ \Rightarrow Y(s) = \frac{1}{(s-1)^4} \]

We recognize that

\[ \frac{1}{(s-1)^4} = \frac{d^3}{ds^3} \left( -\frac{1}{6} \right) \frac{1}{s-1} \]

\[ = \mathcal{L}\left\{ \frac{1}{6} t^3 e^t \right\} \] by using property 1

Hence, \( y(t) = \frac{1}{6} t^3 e^t \) three times

\[ \boxed{5} \]