Homework #2: Green’s functions of Sturm-Liouville problems. Due on Monday, September 29.

Each question is four points. The last one is optional.

1. Consider the following regular Sturm-Liouville problem:
\[-d^2 + u = \lambda u \quad \text{in } (0, \pi),
\]
\[u = 0 \quad \text{on } \{0, \pi\}.
\]
We know that its eigenvalues and eigenfunctions are \( \lambda_i = i^2 \) and \( u_i = \sin ix \), respectively, for \( i = 1, 2, \ldots \). We also know that the set \( \{u_i\}_{i=1}^{\infty} \) is complete in \( L^2(0, \pi) \).
Find the Green’s function \( G(x, s) \) associated to the above boundary-value problem and show that we have
\[G(x, s) = 2 \pi \sum_{i=1}^{\infty} \frac{\sin ix \sin is}{i^2}.
\]
Conclude that we must have
\[\int_0^\pi \int_0^\pi G^2(x, s) \, ds \, dx = \sum_{i=1}^{\infty} \frac{1}{i^4} =: \zeta(4).
\]
Note that \( \zeta \) is nothing but the Riemann zeta function. We have that \( \zeta(4) = \pi^4/90 \).

2. Consider the following regular Sturm-Liouville problem:
\[-d^2 + u = \lambda u \quad \text{in } (0, \pi),
\]
\[\frac{du}{dx} = 0 \quad \text{on } \{0, \pi\},
\]
\[\int_0^\pi u(s) \, ds = 0.
\]
We know that its eigenvalues and eigenfunctions are \( \lambda_i = i^2 \) and \( u_i = \cos ix \), respectively, for \( i = 1, 2, \ldots \). We also know that the set \( \{u_i\}_{i=1}^{\infty} \) is complete in the space of function in \( L^2(0, \pi) \) with zero mean. Find the Green’s function \( G(x, s) \) associated to the above boundary-value problem and show that we have
\[G(x, s) = 2 \pi \sum_{i=1}^{\infty} \frac{\cos ix \cos is}{i^2}.
\]
Conclude that we must have
\[\int_0^\pi \int_0^\pi G^2(x, s) \, ds \, dx = \sum_{i=1}^{\infty} \frac{1}{i^4}.
\]

3. Consider the following singular Sturm-Liouville problem:
\[-\frac{d}{dx} \left((1 - x^2) \frac{d}{dx} u\right) = \lambda u \quad \text{in } (-1, 1),
\]
\[u \text{ and } \frac{du}{dx} \text{ bounded} \quad \text{on } \{-1, 1\},
\]
\[\int_{-1}^1 u(s) \, ds = 0.
\]
We know that its eigenvalues and eigenfunctions are \( \lambda_i = i(i + 1) \) and \( u_i = P_i(x) \) (the Legendre polynomial of degree \( i \)), respectively, for \( i = 1, 2, \ldots \). We also know that the set \( \{u_i\}_{i=1}^{\infty} \) is complete in the space of function in \( L^2(-1, 1) \) with zero mean. Find the Green’s function \( G(x, s) \) associated to the above boundary-value problem and argue that we have

\[
G(x, s) = \sum_{i=1}^{\infty} \frac{2i + 1}{2i(i + 1)} P_i(x) P_i(s).
\]

Conclude that we must have

\[
\int_{-1}^{1} \int_{-1}^{1} G^2(x, s) \, ds \, dx = \sum_{i=1}^{\infty} \frac{1}{i^2(i + 1)^2}.
\]

Verify that the above integral is actually finite.

4. Assume that \( \{u_i\}_{i=1}^{\infty} \) is the set of eigenfunctions of any of the Sturm-Liouville problems of the previous problems; the eigenvalue of \( u_i \) is denoted by \( \lambda_i \). Let us denote by \( V \) the space in which this set is complete.

Prove that, if \( f_N(x) = \sum_{i=1}^{N} a_i u_i(x) \) is the \( L^2 \)-projection of \( f \in V \) into the space spanned by \( \{u_i\}_{i=1}^{N} \), we have that

\[
(f - f_N)(x) = \int_{a}^{b} \left[ G(x, s) - G_N(x, s) \right] \left[ -\frac{d}{ds}(\omega(s) \frac{d}{ds} f(s)) \right] \, ds \quad \text{for } x \in (a, b),
\]

where \( G_N(x, s) = \sum_{i=1}^{N} \frac{u_i(x) u_i(s)}{\lambda_i} \frac{1}{\int_{a}^{b} \omega u_i^2(s)} \). Here \((a, b) := (0, \pi)\) for the first two problems and \((a, b) := (-1, 1)\) for the third. Also \( \omega = 1 \) for the first two problems and \( \omega(s) = 1 - s^2 \) for the third.

Estimate how fast \( \|e_N\|_{L^\infty(a, b)} \) goes to zero in terms of \( N \) for the first two problems assuming that

\[
\|\frac{d}{ds}(\omega \frac{d}{ds} f)\|_{L^2(a, b)} < \infty.
\]

5. Estimate how fast \( \|e_N\|_{L^\infty(a, b)} \) goes to zero in terms of \( N \) for the third problem assuming only that

\[
\|\omega^{1/2} \frac{d}{ds} f\|_{L^2(a, b)} < \infty.
\]