Static Condensation, Hybridization, and the Devising of the HDG Methods

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Abstract In this paper, we review and refine the main ideas for devising the so-called hybridizable discontinuous Galerkin (HDG) methods; we do that in the framework of steady-state diffusion problems. We begin by revisiting the classic techniques of *static condensation* of continuous finite element methods and that of *hybridization* of mixed methods, and show that they can be reinterpreted as discrete versions of a characterization of the associated exact solution in terms of solutions of Dirichlet boundary-value problems on each element of the mesh which are then patched together by transmission conditions across interelement boundaries. We then define the HDG methods associated to this characterization as those using discontinuous Galerkin (DG) methods to approximate the local Dirichlet boundary-value problems, and using weak impositions of the transmission conditions. We give simple conditions guaranteeing the existence and uniqueness of their approximate solutions, and show that, by their very construction, the HDG methods are amenable to static condensation. We do this assuming that the diffusivity tensor can be inverted; we also briefly discuss the case in which it cannot. We then show how a different characterization of the exact solution, gives rise to a different way of statically condensing an already known HDG method. We devote the rest of the paper to establishing bridges between the HDG methods and other methods (the old DG methods, the mixed methods, the staggered DG method and the so-called Weak Galerkin method) and to describing recent efforts for the construction of HDG methods (one for systematically obtaining superconvergent methods and another, quite different, which gives rise to optimally convergent methods). We end by providing a few bibliographical notes and by briefly describing ongoing work.

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1 Introduction

In this paper, we give a short introduction to the devising of the hybridizable discontinuous Galerkin (HDG) in the framework of the following steady-state diffusion model problem:

$$\mathtt{c}\,\boldsymbol{q} + \nabla u = 0 \qquad \text{in } \Omega \subset \mathbb{R}^d, \tag{1a}$$

$$\nabla \cdot \boldsymbol{q} = f \qquad \text{in } \Omega, \tag{1b}$$

$$u = u_D \qquad \text{on } \partial\Omega. \tag{1c}$$

We assume that the data $\mathtt{c}$, f and u_D are smooth functions such that the solution itself is smooth. Here $\mathtt{c}$ is a matrix-valued function which is symmetric and uniformly positive definite on Ω. We are going to closely follow [33], where the HDG methods were introduced.

Since the HDG methods are discontinuous Galerkin (DG) methods, [25], we begin by defining the DG methods for the above boundary-value problem; we follow [3]. Let us first discretize the domain Ω. We denote a triangulation of the domain Ω by $\Omega_h := \{K\}$ and set $\partial\Omega_h := \{\partial K : K \in \Omega_h\}$. The outward unit normal to the element K is denoted by $\boldsymbol{n}$. The set of faces of the element K is denoted by $\mathcal{F}(K)$. An interior face F of the triangulation Ω_h is any set of the form $\partial K^+ \cap \partial K^-$, where $K^\pm$ are elements of Ω_h; we assume that the $(d-1)$-Lebesgue measure of F is not zero. The set of all interior faces is denoted by $\mathcal{F}_h^i$. Similarly, a boundary face F of the triangulation Ω_h is any set of the form $\partial K \cap \partial\Omega$, where K are elements of Ω_h; again, we assume that the $(d-1)$ Lebesgue measure of F is not zero. The set of all boundary faces is denoted by $\mathcal{F}_h^\partial$. The set of interior and boundary faces of the triangulation is denoted by $\mathcal{F}_h$.

The notation associated to the weak formulation of the method is the following. We set

$$(\cdot,\cdot)_{\Omega_h} := \sum_{K\in\Omega_h} (\cdot,\cdot)_K \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\partial\Omega_h} := \sum_{K\in\Omega_h} \langle\cdot,\cdot\rangle_{\partial K},$$

where $(\cdot,\cdot)_K$ denotes the standard $L^2(K)$-inner product, and $\langle\cdot,\cdot\rangle_{\partial K}$ denotes the standard $L^2(\partial K)$-inner product.

We can now introduce the general form of a DG method. The approximate solution $(\boldsymbol{q}_h, u_h)$ given by a DG method is the element of the space $\boldsymbol{V}_h \times W_h$, where

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}(K) \quad \forall K \in \Omega_h\},$$

$$W_h := \{w \in L^2(\Omega) : w|_K \in W(K) \ \forall K \in \Omega_h\}.$$

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satisfying the equations

$$(\mathsf{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0,$$
$$-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h},$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}_h \times W_h$, where the numerical traces $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}$ are approximations to $u|_{\partial\Omega_h}$ and $\boldsymbol{q}\cdot\boldsymbol{n}|_{\partial\Omega_h}$, respectively. The finite dimensional space $\boldsymbol{V}_h \times W_h$ is chosen so that all the integrals in the above weak formulation are well defined.

It remains to discuss how to choose the numerical traces. To do that, let us begin by introducing some useful notation. The traces of the functions ζ and $\boldsymbol{z}$ defined on $K^{\pm} \in \Omega_h$ on the boundary $\partial K^{\pm}$ are denoted by $\zeta^{\pm}$ and $\boldsymbol{z}^{\pm}$, respectively. We use the same notation if the functions ζ and $\boldsymbol{z}$ are defined on $\partial\Omega_h$. Thus, we define the jumps of ζ and $\boldsymbol{z}$ across the interior face $F = \partial K^+ \cap \partial K^-$ by

$$[\![\zeta]\!] := \zeta^+\boldsymbol{n}^+ + \zeta^-\boldsymbol{n}^- \quad \text{and} \quad [\![\boldsymbol{z}]\!] := \boldsymbol{z}^+\cdot\boldsymbol{n}^+ + \boldsymbol{z}^-\cdot\boldsymbol{n}^-,$$

respectively, where $\boldsymbol{n}^{\pm}$ is the outward unit normal to $K^{\pm}$. On boundary faces F, we simply write

$$[\![\zeta]\!] := \zeta\boldsymbol{n} \quad \text{and} \quad [\![\boldsymbol{z}]\!] := \boldsymbol{z}\cdot\boldsymbol{n},$$

with the obvious notation. We say that the numerical traces are *single-valued* if, on $\mathcal{F}_h^i$, $[\![\hat{u}_h]\!] = \boldsymbol{0}$ and $[\![\hat{\boldsymbol{q}}_h]\!] = 0$.

Slightly extending what was done in [3], the numerical traces $\hat{u}_h$ and (the normal component of) $\hat{\boldsymbol{q}}_h$ are linear mappings $\hat{u}_h : \boldsymbol{H}^1(\Omega_h) \times H^1(\Omega_h) \to L^2(\partial\Omega_h)$ $\hat{\boldsymbol{q}}_h : \boldsymbol{H}^1(\Omega_h) \times H^1(\Omega_h) \to \boldsymbol{L}^2(\partial\Omega_h)$ which approximate the traces of u and (the normal component of) $\boldsymbol{q}$ on $\partial\Omega_h$, respectively. We take these numerical traces to be *consistent*. We say that they are consistent if

$$\hat{u}_h(-\mathsf{a}\nabla v, v) = v|_{\partial\Omega_h}, \qquad \hat{\boldsymbol{q}}_h(-\mathsf{a}\nabla v, v)\cdot\boldsymbol{n} = -(\mathsf{a}\nabla v)\cdot\boldsymbol{n}|_{\partial\Omega_h},$$

whenever $[\![\mathsf{a}\nabla v]\!] = 0$ and $[\![v]\!] = \boldsymbol{0}$ on the interior faces $\mathcal{F}_h^i$. Here $\mathsf{a} := \mathsf{c}^{-1}$. This completes the description of the DG methods.

The HDG methods are the DG methods just described which are amenable to *static condensation*. They are thus efficiently implementable and turn out to be more accurate than its predecessors in many instances. None of them fit in the unifying framework developed in [3], since the numerical trace $\hat{u}_h$ of the HDG methods depends on the approximate flux too. The family of DG methods analyzed in [4] includes some HDG methods.

The paper is organized as follows. In Sect. 2, we show that the classic techniques of *static condensation* of continuous finite element methods and that of *hybridization* of mixed methods, introduced back in 1965 in [55] and [52], respectively, can be *reinterpreted* as discrete versions of a characterization of the associated exact solution expressed in terms of solutions of Dirichlet boundary-value problems on each element of the mesh patched together by transmission conditions across interelement boundaries. In Sect. 3, we use this reinterpretation to *define* the HDG methods associated to this characterization as those using discontinuous Galerkin (DG) methods to approximate the local Dirichlet boundary-value problems, and using weak impositions of the transmission conditions. We show that, by construction, the global problem of these HDG methods *only* involves the approximation to the trace of the scalar variable on the faces of the triangulation. We do this assuming that the diffusivity tensor a is invertible; in Sect. 4, we show that it is trivial to treat the case in which it is not. In Sect. 5, we show that a new characterization of the exact solution, based on the elementwise solution of Neumann boundary-value problems, can be used to produce a different type of static condensation of *already known* HDG methods. In Sect. 6, we establish bridges between the HDG and several other methods and comment on two promising ways of devising new HDG methods. We end by providing a few bibliographical notes and by briefly describing ongoing work.

1.1 Note to the Reader

Engineering and Mathematics Graduate Students interested in numerical methods for partial differential equations should be able to read this paper. An elementary background in finite element methods should be enough since here we focus on the ideas guiding the devising of the methods rather than in their rigorous error analyses.

The material of these notes is strongly related to the one presented at the Durham Symposium entitled “Building bridges: Connections and challenges in modern approaches to numerical partial differential equations” at Durham, U.K., July 8–16, 2014, sponsored by the London Mathematical Society, and EPSRC. I would like to express my gratitude to the organizers, especially to G.R. Barrenechea and E. Georgoulis, for the invitation to talk about HDG methods at that meeting.

These notes have evolved from several short courses the author has given: at the Basque Center of Applied Mathematics, Bilbao, Spain, July 9–17, 2009; at the University of Pavia, May 28–June 1, 2012; at the Department of Mathematics & Statistics of the King Fahad University of Petroleum and Minerals, Dec. 2012; at the International Center for Numerical Methods in Engineering, and Universidad Polytecnica de Catalunya, Barcelona, Spain, July 11–15, 2012; at the US National Conference on Computational Mechanics 12, Raleigh, North Carolina, July 22–25, 2013; and at the Department of Mathematics of the Chinese University of Hong Kong, March 19–21, 2014.

2 Static Condensation and Hybridization

Here we argue that the *static condensation* of the continuous Galerkin method, an implementation technique introduced by R.J. Guyan 1965 in [55], can be reinterpreted as a discrete version of a characterization of the exact solution. We also argue that a similar interpretation can be given to the *static condensation* of a mixed method as proposed by Fraejis de Veubeque also in 1965 [52], who showed that this can be achieved provided the mixed method is *hybridized* first. Although the above-mentioned procedures were carried out in the setting of linear elasticity, we present them for our simpler model problem of steady-state diffusion (1).

We proceed as follows. First, we present a characterization of the exact solution in terms of solutions of local problems patched together by means of transmission and boundary conditions. We then show how the original static condensation of the continuous Galerkin method and that of a mixed method can be thought of as discrete versions of such characterization.

2.1 Static Condensation of the Exact Solution

2.1.1 A Characterization of the Exact Solution

Here, for any given triangulation $\Omega_h := \{K\}$ of Ω, we give a characterization of the exact solution in terms of solutions on each of the elements $K \in \Omega_h$, and a single global problem expressed in terms of transmission and boundary conditions.

Suppose that, for each element $K \in \Omega_h$, we define $(\mathbf{Q}, \mathsf{U})$ as the solution of the local problem

$$\begin{aligned} \mathsf{c}\,\mathbf{Q} + \nabla \mathsf{U} &= 0 && \text{in } K, \\ \nabla \cdot \mathbf{Q} &= f && \text{in } K, \\ \mathsf{U} &= \hat{u} && \text{on } \partial K, \end{aligned}$$

where we want the single-valued function $\hat{u}$ to be such that $(\mathbf{Q}, \mathsf{U}) = (\boldsymbol{q}, u)$ on each element $K \in \Omega_h$. We know that this happens *if and only if* $\hat{u}$ enforces the following transmission and boundary conditions:

$$\begin{aligned} [\![\mathbf{Q}]\!] &= 0 && \text{on } F \in \mathcal{F}_h^i, \\ \hat{u} &= u_D && \text{on } F \in \mathcal{F}_h^\partial. \end{aligned}$$

If we now separate the influence of $\hat{u}$ form that of f, we can easily see that we obtained the following result.

Theorem 1 (Characterization of the Exact Solution) *We have that*

$$(\boldsymbol{q}, u) = (\boldsymbol{Q}, \mathsf{U}) = (\boldsymbol{Q}_{\hat{u}}, \mathsf{U}_{\hat{u}}) + (\boldsymbol{Q}_f, \mathsf{U}_f),$$

where, on the element $K \in \Omega_h$, $(\boldsymbol{Q}_{\hat{u}}, \mathsf{U}_{\hat{u}})$ and $(\boldsymbol{Q}_f, \mathsf{U}_f)$ are the solutions of

$$\begin{aligned} \mathsf{c}\,\boldsymbol{Q}_{\hat{u}} + \nabla \mathsf{U}_{\hat{u}} &= 0 \quad \text{in } K, & \mathsf{c}\,\boldsymbol{Q}_f + \nabla \mathsf{U}_f &= 0 \quad \text{in } K,\\ \nabla\cdot\boldsymbol{Q}_{\hat{u}} &= 0 \quad \text{in } K, & \nabla\cdot\boldsymbol{Q}_f &= f \quad \text{in } K,\\ \mathsf{U}_{\hat{u}} &= \hat{u} \quad \text{on } \partial K, & \mathsf{U}_f &= 0 \quad \text{on } \partial K, \end{aligned}$$

and where $\hat{u}$ is the single-valued function solution of

$$\begin{aligned} -[\![\boldsymbol{Q}_{\hat{u}}]\!] &= [\![\boldsymbol{Q}_f]\!] \quad \text{if } F \in \mathcal{F}_h^i,\\ \hat{u} &= u_D \quad \text{if } F \in \mathcal{F}_h^\partial. \end{aligned}$$

2.1.2 An Example

Let us illustrate this result with a simple but revealing case. Take $\Omega := (0, 1)$ with $K = (x_{i-1}, x_i)$ for $i = 1, \ldots, N$ where $x_0 = 0$ and $x_N = 1$. For simplicity, we take c to be a constant. We then have that

$$(\boldsymbol{q}, u) = (\mathbf{Q}_{\hat{u}}, \mathsf{U}_{\hat{u}}) + (\mathbf{Q}_f, \mathsf{U}_f),$$

where, for $i = 1, \ldots, N$, the functions $(\mathbf{Q}_{\hat{u}}, \mathsf{U}_{\hat{u}})$ and $(\mathbf{Q}_f, \mathsf{U}_f)$ are the solutions of the local problem

$$\begin{aligned} \mathsf{c}\,\mathbf{Q}_{\hat{u}} + \frac{d}{dx}\mathsf{U}_{\hat{u}} &= 0 \quad \text{in } (x_{i-1}, x_i), & \mathsf{c}\,\mathbf{Q}_f + \frac{d}{dx}\mathsf{U}_f &= 0 \quad \text{in } (x_{i-1}, x_i),\\ \frac{d}{dx}\mathbf{Q}_{\hat{u}} &= 0 \quad \text{in } (x_{i-1}, x_i), & \frac{d}{dx}\mathbf{Q}_f &= f \quad \text{in } (x_{i-1}, x_i),\\ \mathsf{U}_{\hat{u}} &= \hat{u} \quad \text{on } \{x_{i-1}, x_i\}, & \mathsf{U}_f &= 0 \quad \text{on } \{x_{i-1}, x_i\}. \end{aligned}$$

Note that we still do not know the actual values of the function $\hat{u} : \{x_i\}_{i=0}^N \mapsto \mathbb{R}$, but once we obtain them, we can readily get the exact solution $(\boldsymbol{q}, u)$. To find those values, we only have to solve the global problem

$$\begin{aligned} -\mathbf{Q}_{\hat{u}}(x_i^-) + \mathbf{Q}_{\hat{u}}(x_i^+) &= \mathbf{Q}_f(x_i^-) - \mathbf{Q}_f(x_i^+) \quad \text{for } i = 1, \ldots, N-1,\\ \hat{u}(x_i) &= u_D(x_i) \quad \text{for } i = 0, N. \end{aligned}$$

Now, let us solve the local problems and then find the global problem. A simple computation gives that the solutions of the local problems are

$$\mathbf{Q}_{\hat{u}}(x) = -\frac{\hat{u}_i - \hat{u}_{i-1}}{\mathsf{c}\, h_i}, \qquad \mathbf{Q}_f(x) = -\mathsf{c}^{-1}\int_{x_{i-1}}^{x_i} G_x^i(x,s) f(s)\, ds,$$

$$\mathsf{U}_{\hat{u}}(x) = \varphi_i(x)\,\hat{u}_i + \varphi_{i-1}(x)\,\hat{u}_{i-1}, \qquad \mathsf{U}_f(x) = \int_{x_{i-1}}^{x_i} G^i(x,s) f(s)\, ds,$$

where $h_i := x_i - x_{i-1}$ and G^i is the Green's function of the second local problem, namely,

$$G^i(x,s) := \begin{cases} \mathsf{c}\, h_i\, \varphi_i(s)\, \varphi_{i-1}(x) & \text{if } x_{i-1} \leqslant s \leqslant x, \\ \mathsf{c}\, h_i\, \varphi_i(x)\, \varphi_{i-1}(s) & \text{if } x \leqslant s \leqslant x_i. \end{cases}$$

where

$$\varphi_i(s) := \begin{cases} (s - x_{i-1})/h_i & \text{if } x_{i-1} \leqslant s \leqslant x_i, \\ (x_{i+1} - s)/h_{i+1} & \text{if } x_i \leqslant s \leqslant x_{i+1}. \end{cases}$$

As a consequence, the global problem for the values $\{\hat{u}_i\}_{i=0}^N$ is

$$\frac{\hat{u}_i - \hat{u}_{i-1}}{\mathsf{c}\, h_i} - \frac{\hat{u}_{i+1} - \hat{u}_i}{\mathsf{c}\, h_{i+1}} = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(s) f(s)\, ds \qquad \text{for } i = 1, \ldots, N-1,$$

$$\hat{u}_j = u_D(x_j) \qquad \text{for } j = 0, N.$$

In other words, the values of the exact solution at the nodes of the triangulation, $\{\hat{u}_i\}_{i=0}^N$, can be obtained by inverting a (symmetric positive definite) tridiagonal matrix of order $N + 1$.

2.2 Static Condensation of the Continuous Galerkin Method

Now, we show that a characterization of the continuous Galerkin method similar to that one just obtained for the exact solution can be interpreted as the original static condensation of the method [55].

2.2.1 A Characterization of the Approximate Solution

The continuous Galerkin method provides an approximation to u, u_h, in the space

$$W_h = \{w \in \mathcal{C}^0(\Omega) : w|_K \in W(K)\ \forall K \in \Omega_h\}.$$

It determines it by requiring that it be the only solution in $W_h(u_D)$ of the equation

$$(\mathsf{a}\,\nabla u_h, \nabla w)_\Omega = (f, w)_\Omega \quad \forall w \in W_h(0).$$

where $W_h(g) = \{w \in W_h : w = I_h(g) \text{ on } \partial\Omega\}$, and I_h is a suitably defined interpolation operator.

Now, to obtain our characterization of the approximate solution, we need to split the spaces in a suitable manner. Thus, for each element $K \in \Omega_h$, we define the space associated to the *interior* degrees of freedom,

$$W_0(K) := \{w \in W(K) : w|_{\partial K} = 0\},$$

and the space associated to the degrees of freedom on the *boundary*,

$$W_\partial(K) := \{w \in W(K) : w|_{\partial K} = 0 \implies w|_K = 0\}.$$

Clearly, $W(K) = W_0(K) + W_\partial(K)$ for all $K \in \Omega_h$, and so $W_h = W_{0,h} + W_{\mathcal{F}_h}$ where

$$W_{0,h} := \{w \in W_h : w|_K \in W_0(K)\ \forall K \in \Omega_h\},$$
$$W_{\mathcal{F}_h} := \{w \in W_h : w|_K \in W_\partial(K)\ \forall K \in \Omega_h\}.$$

We also need to introduce the following sets of traces on $\mathcal{F}_h$:

$$M_h := \{w|_{\mathcal{F}_h} : w \in W_h\},$$
$$M_h(g) := \{\mu \in M_h : \mu|_{\partial\Omega} = I_h(g)\}.$$

Note that the trace into $\mathcal{F}_h$ is an isomorphism between $W_{\mathcal{F}_h}$ and M_h.

Suppose that, for each element $K \in \Omega_h$, we define $\mathsf{U} \in W(K)$ as the solution of the local problem

$$(\mathsf{a}\,\nabla \mathsf{U}, \nabla w)_K = (f, w)_K \quad \forall w \in W_0(K),$$
$$\mathsf{U} = \hat{u}_h \quad \text{on } \partial K,$$

where we want to chose the function $\hat{u}_h \in M_h$ in such a way that $\mathsf{U} = u_h$ on each element $K \in \Omega_h$. This happens if and only $\hat{u}_h$ is such that

$$(\mathsf{a}\,\nabla \mathsf{U}, \nabla w)_\Omega = (f, w)_\Omega \quad \forall w \in W_{\mathcal{F}_h},$$
$$\hat{u}_h = I_h(u_D) \quad \text{on } \partial\Omega.$$

If we separate the influence of $\hat{u}_h$ from that of f in the definition of the local problems, and rework the formulation of the global problem, we get the following result.

Theorem 2 (Characterization of the Continuous Galerkin Method) *The approximation given by the continuous Galerkin method can be written as*

$$u_h = \mathsf{U} = \mathsf{U}_{\hat{u}_h} + \mathsf{U}_f,$$

where, on the element $K \in \Omega_h$, $\mathsf{U}_{\hat{u}_h}$ *and* U_f *are the elements of* $W(K)$ *that solve the local problems*

$$\begin{aligned}(\mathsf{a}\,\nabla\mathsf{U}_{\hat{u}_h}, \nabla w)_K &= 0 \quad \forall w \in W_0(K) & (\mathsf{a}\,\nabla\mathsf{U}_f, \nabla w)_K &= (f, w)_K \quad \forall w \in W_0(K),\\ \mathsf{U}_{\hat{u}_h} &= \hat{u}_h \quad \text{on } \partial K & \mathsf{U}_f &= 0 \qquad\quad \text{on } \partial K,\end{aligned}$$

and $\hat{u}_h$ *is the element of* $M_h(u_D)$ *that solves the global problem*

$$(\mathsf{a}\,\nabla\mathsf{U}_{\hat{u}_h}, \nabla\mathsf{U}_\mu)_\Omega = (f, \mathsf{U}_\mu)_\Omega \qquad \forall\ \mu \in M_h(0).$$

Note that, although the static condensation [55] is carried out directly on the stiffness matrix of the method, this result shows how to use (local and global) weak formulations to achieve exactly the same thing.

Proof By the linearity of the problem, we only have to justify the characterization of the function $\hat{u}_h$. Let us start from the fact that $\hat{u}_h$ is the element of $M_h(u_D)$ which solves the global problem

$$(\mathsf{a}\,\nabla\mathsf{U}_{\hat{u}_h}, \nabla w)_\Omega + (\mathsf{a}\,\nabla\mathsf{U}_f, \nabla w)_\Omega = (f, w)_\Omega \qquad \forall\ w \in W_{\mathcal{F}_h}.$$

Now, note that, for any $w \in W_h$, we can define the function w_0 by the equation

$$w = \mathsf{U}_\mu + w_0,$$

where $\mu := w|_{\mathcal{F}_h}$; this readily implies that $w_0 \in W_{0,h}$. If we now insert this expression in the equation and take into consideration the definition of the solution of the local problems, that is, that

$$\begin{aligned}(\mathsf{a}\,\nabla\mathsf{U}_{\hat{u}_h}, \nabla w_0)_\Omega &= 0,\\ (\mathsf{a}\,\nabla\mathsf{U}_f, \nabla\mathsf{U}_\mu)_\Omega &= 0,\\ (\mathsf{a}\,\nabla\mathsf{U}_f, \nabla w_0)_\Omega &= (f, w_0)_\Omega,\end{aligned}$$

we finally get the wanted formulation. This completes the proof. □

2.2.2 The Numerical Trace of the Flux

A quick comparison of the above result with the one for the exact solution, suggests that the global problem for the continuous Galerkin method is a *transmission* condition on a discrete version of the normal component of the flux. This little known fact will allow us to *identify* the numerical trace of the approximate flux for the continuous Galerkin method.

To do this, we first write the global problem in its original form, that is,

$$(\mathsf{a}\,\nabla u_h, \nabla w)_\Omega = (f, w)_\Omega \qquad \forall\, w \in W_{\mathcal{F}_h},$$

and perform a simple integration by parts to get

$$-(\nabla\cdot(\mathsf{a}\,\nabla u_h), w)_{\Omega_h} + \langle(\mathsf{a}\,\nabla u_h)\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} = (f, w)_\Omega \qquad \forall\, w \in W_{\mathcal{F}_h},$$

Let us now define, for each element $K \in \Omega_h$, the function $R_h \in W_\partial(K)$ satisfying the equation

$$\langle R_h, w\rangle_{\partial K} = (\nabla\cdot(\mathsf{a}\,\nabla u_h) + f, w)_K \qquad \forall w \in W_\partial(K).$$

Thus, the function R_h is a projection of the *residual* $\nabla\cdot(\mathsf{a}\,\nabla u_h) + f$. With this definition, we get that

$$\langle(-\mathsf{a}\,\nabla u_h)\cdot\boldsymbol{n} + R_h, w\rangle_{\partial\Omega_h} = 0 \qquad \forall\, w \in W_{\mathcal{F}_h},$$

which can be interpreted as a transmission condition forcing the normal component of numerical trace of the flux

$$\hat{\boldsymbol{q}}_h\cdot\boldsymbol{n} := (-\mathsf{a}\,\nabla u_h)\cdot\boldsymbol{n} + R_h \quad \text{on } \partial\Omega_h,$$

to be weakly continuous across interelement boundaries.

2.2.3 Relation with Static Condensation

Let us now show that this characterization is nothing but an application of the well-known technique of *static condensation* [55]. Static condensation was conceived as a way to reducing the size of the stiffness matrix. Indeed, if $[u_h]$ is the vector of degrees of freedom of the approximation u_h, and the matrix equation of the continuous Galerkin method is

$$K\,[u_h] = [f],$$

the static condensation consists in partitioning the vector of degrees of freedom $[u_h]$ into two smaller vectors, namely, the degrees of freedom interior to the elements, $[\mathsf{U}]$, and the degrees of freedom associated to the boundaries of the elements,

$[\hat{u}_h]$, and then *eliminating* $[\mathsf{U}]$ from the equations. Indeed, taking into account this partition, the above equation reads

$$\begin{bmatrix} K_{00} & K_{0\partial} \\ K_{\partial 0} & K_{\partial\partial} \end{bmatrix} \begin{bmatrix} [\mathsf{U}] \\ [\hat{u}_h] \end{bmatrix} = \begin{bmatrix} f_0 \\ f_\partial \end{bmatrix}.$$

By our choice of the degrees of freedom, the matrix K_{00} is easy to invert since it is *block diagonal*, each block being associated to a local problem. We thus get

$$[\mathsf{U}] = -K_{00}^{-1} K_{0\partial}[\hat{u}_h] + K_{00}^{-1}[f_0].$$

We can now eliminate $[\mathsf{U}]$ from the original matrix equation to obtain

$$(-K_{\partial 0}\, K_{00}^{-1}\, K_{0\partial} + K_{\partial\partial})[\hat{u}_h] = -K_{\partial 0}\, K_{00}^{-1}[f_0] + [f_\partial].$$

The matrix in the left-hand side, nowadays called the Schur complement of the matrix K_{00}, is clearly smaller than the original matrix K and is also easier to numerically invert. We have thus shown that our characterization of the approximate solution of the continuous Galerkin method is nothing but another way of carrying out the good, old static condensation. The former expresses in terms of weak formulations what the latter does directly on the matrix equations itself.

2.2.4 An Example

Let us now illustrate this procedure in our simple one-dimensional example. We take

$$W(K) := \mathcal{P}_k(K),$$

where $\mathcal{P}_k(K)$ denotes the space of polynomials of degree at most k defined on the set K. We begin by solving the local problems. If we use the notation $\hat{u}_i = \hat{u}_h(x_i)$ for $i = 0, \ldots, N$, a few manipulations (and the proper choice of the basis functions) allow us to see that the solutions of the local problems are

$$\mathsf{U}_{\hat{u}}(x) = \varphi_i(x)\,\hat{u}_i + \varphi_{i-1}(x)\,\hat{u}_{i-1} \qquad \mathsf{U}_f(x) = \int_{x_{i-1}}^{x_i} G_h^i(x,s) f(s)\, ds,$$

where $h_i := x_i - x_{i-1}$ and G_h^i is the discrete Green's function of the second local problem, namely,

$$G_h^i(x,s) := \frac{h_i}{4\mathsf{a}} \sum_{\ell=1}^{k-1} \frac{1}{2\ell+1}\,(P_{\ell+1}^i - P_{\ell-1}^i)(x)\,(P_{\ell+1}^i - P_{\ell-1}^i)(s)$$

where $P_n^i(x) := P_n(T^i(x))$, $T^i(\zeta) := (\zeta - (x_i + x_{i-1})/2)/(h_i/2)$ and P_n is the Legendre polynomial of degree n. As a consequence, the global problem for the values $\{\hat{u}_i\}_{i=0}^N$ is

$$\begin{aligned}
\mathsf{a}\,\frac{\hat{u}_i - \hat{u}_{i-1}}{h_i} - \mathsf{a}\,\frac{\hat{u}_{i+1} - \hat{u}_i}{h_{i+1}} &= \int_{x_{i-1}}^{x_{i+1}} \varphi_i(s) f(s)\, ds \qquad \text{for } i = 1, \dots, N-1,\\
\hat{u}_j &= u_D(x_j) \qquad\qquad\qquad\quad \text{for } j = 0, N.
\end{aligned}$$

Note that the size of the matrix equation of the global problem is *independent* of the value of the polynomial degree k, a reflection of the effectiveness of the static condensation technique. Note also that the values of the approximate solution at the nodes of the triangulation, $\{\hat{u}_i\}_{i=0}^N$, are actually exact, as expected.

2.3 Static Condensation of Mixed Methods by Hybridization

Next, we show how to extend what was done for the continuous Galerkin method to mixed methods. A particular important point we want to emphasize here is that *hybridization* of a mixed method is what *allows* it to be statically condensed, as first realized in [52].

2.3.1 A Characterization of the Approximate Solution

A mixed method seeks approximations to the flux $\boldsymbol{q} := -\mathsf{a}\nabla u$, $\boldsymbol{q}_h$, and the scalar u, u_h, in the finite dimensional spaces

$$\begin{aligned}
\boldsymbol{\mathcal{V}}_h &= \{\boldsymbol{v} \in \boldsymbol{H}(div, \Omega) : \boldsymbol{v}|_K \in \boldsymbol{V}(K) \quad \forall K \in \Omega_h\}.\\
W_h &= \{w \in L^2(\Omega) : \qquad w|_K \in W(K)\ \forall K \in \Omega_h\},
\end{aligned}$$

respectively. It determines the function $(\boldsymbol{q}_h, u_h)$ as the only element of $\boldsymbol{\mathcal{V}}_h \times W_h$ satisfying the equations

$$\begin{aligned}
(\mathsf{c}\,\boldsymbol{q}_h, \boldsymbol{v})_\Omega - (u_h, \nabla\cdot\boldsymbol{v})_\Omega &= -\langle u_D, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega} \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h,\\
(\nabla\cdot\boldsymbol{q}_h, w)_\Omega &= (f, w)_\Omega \qquad\qquad \forall w \in W_h.
\end{aligned}$$

For mixed methods, the choice of the finite dimensional space $\boldsymbol{\mathcal{V}}_h \times W_h$ is not simple, but here we assume that *it has* been properly chosen as to define a unique approximate solution.

Now, suppose that, for each element $K \in \Omega_h$, we define $(\mathbf{Q}, \mathsf{U}) \in \boldsymbol{V}(K) \times W(K)$ as the solution of the local problem

$$(\mathsf{c}\,\mathbf{Q}, \boldsymbol{v})_K - (\mathsf{U}, \nabla \cdot \boldsymbol{v})_K = \langle \hat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} \quad \forall \boldsymbol{v} \in \boldsymbol{V}(K),$$
$$(\nabla \cdot \mathbf{Q}, w)_K = (f, w)_K \qquad \forall w \in W(K).$$

This problem is well defined since it is nothing but the application of the mixed method, which we assume to be well defined, to the single element $K \in \Omega_h$. As before, we want to choose the function $\hat{u}_h$ in some finite dimensional space M_h in such a way that $(\mathbf{Q}, \mathsf{U}) = (\boldsymbol{q}_h, u_h)$ on each element $K \in \Omega_h$. For this to hold, we only need to guarantee that

$$\mathbf{Q} \in \boldsymbol{\mathcal{V}}_h,$$
$$\langle \hat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega} = \langle u_D, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega} \quad \forall\ \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h.$$

The first property is a transmission condition since it holds if and only if the normal component of $\mathbf{Q}$ is continuous across interelement boundaries. The second condition is nothing but a weak form of the Dirichlet boundary condition.

As for the case of the continuous Galerkin method, the choice of the space M_h has to be made in such a way that the above two conditions do *determine* the numerical trace $\hat{u}_h$. Typically, we take

$$M_h := \{\mu \in L^2(\mathcal{F}_h) :\ \exists\ \boldsymbol{v} \in \boldsymbol{V}_h : \mu = [\![\boldsymbol{v}]\!] \text{ on } \mathcal{F}_h\}.$$

Thus, if we set $M_h(g) := \{\mu \in M_h :\ \langle \mu, \eta \rangle_{\partial\Omega} = \langle g, \eta \rangle_{\partial\Omega}\ \forall \eta \in M_h\}$, the global problem can be expressed as follows:

$$\langle \mathbf{Q} \cdot \boldsymbol{n}, \mu \rangle_{\partial\Omega_h} = 0 \qquad \forall\ \mu \in M_h(0),$$
$$\hat{u}_h \in M_h(u_D).$$

Indeed, note that, for any $\mu \in M_h(0)$,

$$\langle \mathbf{Q} \cdot \boldsymbol{n}, \mu \rangle_{\partial\Omega_h} = \langle \mathbf{Q}, \mu \rangle_{\partial\Omega_h \setminus \partial\Omega} = \langle\, [\![\mathbf{Q}]\!], \mu \rangle_{\mathcal{F}_h^i},$$

and if this quantity is zero, we certainly have that $\mathbf{Q} \in \boldsymbol{\mathcal{V}}_h$, as wanted. So, let us assume then that the above global problem for $\hat{u} \in M_h$ is well defined.

So, we have obtained the following result.

Theorem 3 (Characterization of the Mixed Method) *The solution of the mixed method can be written as*

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{Q}, \mathsf{U}) = (\boldsymbol{Q}_{\hat{u}_h}, \mathsf{U}_{\hat{u}_h}) + (\boldsymbol{Q}_f, \mathsf{U}_f),$$

where, on each element $K \in \Omega_h$, for any $\mu \in L^2(\partial K)$ and $f \in L^2(K)$, the functions $(\boldsymbol{Q}_\mu, \mathsf{U}_\mu)$ and $(\boldsymbol{Q}_f, \mathsf{U}_f)$ are the elements of $\boldsymbol{V}(K) \times W(K)$ which solve the local problems

$$(\mathsf{c}\,\boldsymbol{Q}_\mu, \boldsymbol{v})_K - (\mathsf{U}_\mu, \nabla\cdot\boldsymbol{v})_K = -\langle \mu, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K}, \qquad (\mathsf{c}\,\boldsymbol{Q}_f, \boldsymbol{v})_K - (\mathsf{U}_f, \nabla\cdot\boldsymbol{v})_K = 0,$$
$$(\nabla\cdot\boldsymbol{Q}_\mu, w)_K = 0, \qquad (\nabla\cdot\boldsymbol{Q}_f, w)_K = (f, w)_K,$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K) \times W(K)$, and the function $\hat{u}_h$ is the element of $M_h(u_D)$ which solves the global problem

$$(\mathsf{c}\,\boldsymbol{Q}_{\hat{u}_h}, \boldsymbol{Q}_\mu)_{\Omega_h} = (f, \mathsf{U}_\mu)_{\Omega_h} \quad \forall\ \mu \in M_h(0),$$

Proof We only have to prove that $\hat{u}_h \in M_h(u_D)$ satisfies the equation

$$-\langle \mathbf{Q}_{\hat{u}_h}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} = \langle \mathbf{Q}_f\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} \quad \forall\ \mu \in M_h(0).$$

But, by the definition of the local problems, we have

$$\begin{aligned}
-\langle \mathbf{Q}_{\hat{u}_h}\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} &= (\mathsf{c}\,\mathbf{Q}_\mu, \mathbf{Q}_{\hat{u}_h})_{\Omega_h},\\
\langle \mathbf{Q}_f\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} &= -(\mathsf{c}\,\mathbf{Q}_\mu, \mathbf{Q}_f)_{\Omega_h} + (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_f)_{\Omega_h}\\
&= -(\mathsf{U}_f, \nabla\cdot\mathbf{Q}_{\hat{u}_h})_{\Omega_h} + (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_f)_{\Omega_h}\\
&= (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_f)_{\Omega_h}\\
&= (f, \mathsf{U}_\mu)_{\Omega_h},
\end{aligned}$$

and the identity follows. This completes the proof. □

2.3.2 Relation with Static Condensation and Hybridization

Let us now show that what we have done is nothing but the *static condensation* of the *hybridized* version of the mixed method as done by Fraejis de Veubeke in [52]. Suppose that the matrix equation of the mixed method reads

$$\begin{bmatrix} \mathcal{A} & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} [\boldsymbol{q}_h] \\ [u_h] \end{bmatrix} = \begin{bmatrix} [u_D] \\ [f] \end{bmatrix}.$$

It is not easy to eliminate $[\boldsymbol{q}_h]$ from this equation since the matrix $\mathcal{A}$ is *not* block diagonal because, since $\boldsymbol{q}_h \in \mathcal{V}_{\boldsymbol{h}}$, its normal component is continuous across inter element boundaries. To overcome this unwanted feature, Fraejis de Veubeque relaxed the continuity condition on $\boldsymbol{q}_h$ and worked with a totally discontinuous approximation $\mathbf{Q}$ instead. Because of this, he had to introduce the *hybrid* unknown $\hat{u}_h$, an approximation to the trace of u on each element; this is why this procedure

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receives the name of *hybridization* of the mixed method. Finally, in order to guarantee that $\mathbf{Q}$ be identical to the original function $\boldsymbol{q}_h$, he then forced it to have a continuous normal component at the interelement boundaries. This operation resulted the following matrix equation:

$$\begin{bmatrix} A & B & C \\ B^t & 0 & 0 \\ C^t & 0 & 0 \end{bmatrix} \begin{bmatrix} [\mathbf{Q}] \\ [\mathsf{U}] \\ [\hat{u}_h] \end{bmatrix} = \begin{bmatrix} -C_\partial[u_D] \\ [f] \\ 0 \end{bmatrix}.$$

Here, $[\hat{u}_h]$ denotes the digressive freedom of the function $\hat{u}_h$ restricted to the interior faces. On the boundary faces, the relation of $\hat{u}_h$ to u_D is already captured by the right-hand side of the first equation. Note that, since the first two equations define the local problems, we can easily solve them to obtain

$$\begin{bmatrix} [\mathbf{Q}] \\ [\mathsf{U}] \end{bmatrix} = \begin{bmatrix} A & B \\ B^t & 0 \end{bmatrix}^{-1} \begin{bmatrix} -C[\hat{u}_h] - C_\partial[u_D] \\ [f] \end{bmatrix}.$$

The third equation, $C\,[\mathbf{Q}] = 0$ enforces the continuity of the normal component of $\mathbf{Q}$ across inter element boundaries; it is this equation that determines the hybrid unknown in the interior faces, $[\hat{u}_h]$. A few computations show that the resulting matrix equation is of the form

$$H[\,\hat{u}\,] = H_\partial[\,u_D\,] + J\,[\,f\,], \qquad H := C^t E\, C, \qquad E := A^{-1} - A^{-1}B\,(B^t A^{-1} B)^{-1}\, B^t A^{-1},$$

and we see that, as expected, the matrix H is symmetric. Moreover, H is positive definite and E is block-diagonal.

2.3.3 An Example

Next, let us illustrate this procedure in our simple one-dimensional example. We take

$$V(K) \times W(K) := \mathcal{P}_{k+1}(K) \times \mathcal{P}_k(K).$$

We begin by solving the local problems. A little computation gives that the solutions of the local problems are

$$\mathbf{Q}_{\hat{u}}(x) = -\frac{\hat{u}_i - \hat{u}_{i-1}}{\mathsf{c}\, h_i}, \qquad \mathbf{Q}_f(x) = \int_{x_{i-1}}^{x_i} H_h^i(x,s) f(s)\, ds,$$

$$\mathsf{U}_{\hat{u}}(x) = \varphi_i(x)\,\hat{u}_i + \varphi_{i-1}(x)\,\hat{u}_{i-1}, \qquad \mathsf{U}_f(x) = \int_{x_{i-1}}^{x_i} G_h^i(x,s) f(s)\, ds,$$

where $h_i := x_i - x_{i-1}$ and

$$H_h^i(x,s) := \varphi_i(x)\,\varphi_i(s) - \varphi_{i-1}(x)\,\varphi_{i-1}(s) + \frac{1}{2}\sum_{\ell=1}^{k}(P_{\ell+1}^i - P_{\ell-1}^i)(x)\,P_\ell^i(s),$$

$$G_h^i(x,s) := \frac{\mathsf{c}\,h_i}{4}\sum_{\ell=1}^{k-1}\frac{1}{2\ell+1}\,(P_{\ell+1}^i - P_{\ell-1}^i)(x)\,(P_{\ell+1}^i - P_{\ell-1}^i)(s).$$

Let us recall that $P_n^i(x) := P_n(T^i(x))$, $T^i(\zeta) := (\zeta - (x_i + x_{i-1})/2)/(h_i/2)$ and P_n is the Legendre polynomial of degree n. Note that the function G_h^i approximates the Green function G^i whereas $-\mathsf{c}\,H_h^i$ approximates its partial derivative G_x^i. As a consequence, the global problem for the values $\{\hat{u}_i\}_{i=0}^N$ is

$$\frac{\hat{u}_i - \hat{u}_{i-1}}{\mathsf{c}\,h_i} - \frac{\hat{u}_{i+1} - \hat{u}_i}{\mathsf{c}\,h_{i+1}} = \int_{x_{i-1}}^{x_{i+1}} \varphi_i(s) f(s)\,ds \qquad \text{for } i = 1,\ldots,N-1,$$

$$\hat{u}_j = u_D(x_j) \qquad \text{for } j = 0, N.$$

We thus see that the values of the approximate solution at the nodes of the triangulation, $\{\hat{u}_i\}_{i=0}^N$, are actually exact, as expected.

3 HDG Methods

In this section, we show how to use a discrete version of the characterization of the exact solution obtained in the previous section to devise HDG methods for our model problem (1). The local problems are solved by using a DG method and the transmission conditions by a simple weak formulation. As a consequence, the resulting HDG methods are DG methods whose distinctive feature is that they are amenable to hybridization and hence to static condensation. Let us emphasize that this does not happen by accident, but because they are constructed by using a discrete version of the characterization of the exact solution worked out in the previous section.

After defining the HDG methods, we establish a simple result about the existence and uniqueness of their approximate solution and display some examples. We end by showing several different ways of presenting them which will be useful for relating them to other numerical methods.

We follow closely the work done in 2009 [33] for the original HDG methods, as well as the work done in the 2014 review paper [23] for HDG methods for the Stokes system of incompressible fluid flow.

3.1 Definition

We take the approximate solution of the HDG methods to be the function

$$(\boldsymbol{q}_h, u_h) = (\mathbf{Q}, \mathsf{U}),$$

where, on the element $K \in \Omega_h$, $(\mathbf{Q}, \mathsf{U}) \in \boldsymbol{V}(K) \times W(K)$ is the solution of the local problem

$$\begin{aligned}(\mathsf{c}\,\mathbf{Q}, \boldsymbol{v})_K - (\mathsf{U}, \nabla\cdot\boldsymbol{v})_K + \langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\ -(\mathbf{Q}, \nabla w)_K + \langle \hat{\mathbf{Q}}\cdot\boldsymbol{n}, w\rangle_{\partial K} &= (f, w)_K \quad \forall w \in W(K),\end{aligned}$$

where the numerical trace $\hat{\mathbf{Q}}$ has to be suitably chosen. Ideally, the numerical trace of the flux $\hat{\mathbf{Q}}$ should be chosen so that it

1. is consistent,
2. only depends (linearly) on $\mathbf{Q}|_K, \mathsf{U}|_K$ and $\hat{u}_h|_{\partial K}$,
3. renders the local problem solvable.

Our *favorite* choice is

$$\hat{\mathbf{Q}}\cdot\boldsymbol{n} := \mathbf{Q}\cdot\boldsymbol{n} + \tau(\mathsf{U} - \hat{u}_h) \quad \text{on } \partial K,$$

where the function τ is linear. We are also going to require that τ be *symmetric*, that is, that, for all $K \in \Omega_h$,

$$\langle \tau(w), \omega\rangle_{\partial K} = \langle w, \tau(\omega)\rangle_{\partial K} \quad \forall\, w, \omega \in W(K) + M_h(\partial K).$$

Although there are many other choices, we are going to use this one from now on; not only it is very natural but it actually covers *all* the known HDG methods.

To complete the definition of the HDG methods, we take the function $\hat{u}_h$ in the space

$$M_h := \{\mu \in L^2(\mathcal{F}_h) :\ \mu_F \in M(F)\ \forall F \in \mathcal{F}_h\},$$

where $M(F)$ is a suitably chosen finite dimensional space, and require that it be determined as the solution of the following weakly imposed transmission and boundary conditions:

$$\begin{aligned}&\langle \mu,\, [\![\hat{\mathbf{Q}}]\!]\rangle_{\mathcal{F}_h^i} = \langle \mu, \hat{\mathbf{Q}}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h\setminus\partial\Omega} = 0,\\ &\langle \mu, \hat{u}_h\rangle_{\partial\Omega} = \langle \mu, u_D\rangle_{\partial\Omega},\end{aligned}$$

for all $\mu \in M_h$. This completes the definition of the HDG methods.

The HDG methods are obtained by choosing different functions τ and different local spaces $\boldsymbol{V}(K)$, $W(K)$ and $M(F)$.

3.2 Existence and Uniqueness

We now provide simple conditions on the local spaces and the function τ ensuring, not only that the local problems are solvable, but that the global problem is also well posed. To do that, we use an *energy* identity we obtain next which will also shed light on the role to the function τ.

Proposition 1 (The Local Energy Identity) *For any element $K \in \Omega_h$, we have*

$$(\mathsf{c}\,\boldsymbol{Q},\boldsymbol{Q})_K + \langle(\mathsf{U}-\hat{u}_h),\tau(\mathsf{U}-\hat{u}_h)\rangle_{\partial K} = (f,\mathsf{U})_K - \langle \hat{u}_h, \hat{\boldsymbol{Q}}\cdot\boldsymbol{n}\rangle_{\partial K}.$$

Note that the exact solution satisfies the following energy identity:

$$(\mathsf{c}\,\boldsymbol{q},\boldsymbol{q})_K = (f,u)_K - \langle u, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial K}.$$

Typically, the terms $(\mathsf{c}\,\boldsymbol{q},\boldsymbol{q})_K$ and $(\mathsf{c}\,\mathbf{Q},\mathbf{Q})_K$ are interpreted as the energy stored inside the element K. It is thus reasonable to interpret the term $\langle(\mathsf{U}-\hat{u}_h),\tau(\mathsf{U}-\hat{u}_h)\rangle_{\partial K}$ as an *energy* associated with the jumps $\mathsf{U}-\hat{u}_h$ at the boundary of the element ∂K. Since all energies are nonnegative, we assume that the function τ is such that

$$\langle\tau(w-\mu), w-\mu\rangle_{\partial K} \geqslant 0 \qquad \forall (w,\mu)\in W(K)\times M(\partial K), \tag{2a}$$

where

$$M(\partial K) := \{\mu \in L^2(\partial K) :\ \mu|_F \in M(F),\ \text{for any face } F \in \mathcal{F}_h \text{ lying on } \partial K\}. \tag{2b}$$

We now see that the role of τ is to transform the discrepancy between U and $\hat{u}_h$ on ∂K into an energy. Since an increase of energy is typically associated with an enhancement of the stability properties of the numerical method, τ is called the *stabilization* function.

Let us now prove Proposition 1.

Proof If we take $(\boldsymbol{v}, w) := (\mathbf{Q},\mathsf{U})$ in the equations of the local problems, we get

$$(\mathsf{c}\,\mathbf{Q},\mathbf{Q})_K - (\mathsf{U},\nabla\cdot\mathbf{Q})_K + \langle \hat{u}_h, \mathbf{Q}\cdot\boldsymbol{n}\rangle_{\partial K} = 0,$$
$$-(\mathbf{Q},\nabla\mathsf{U})_K + \langle \hat{\mathbf{Q}}\cdot\boldsymbol{n},\mathsf{U}\rangle_{\partial K} = (f,\mathsf{U})_K,$$

and adding the two equations, we obtain

$$(\mathsf{c}\,\mathbf{Q},\mathbf{Q})_K + \langle(\hat{\mathbf{Q}}-\mathbf{Q})\cdot\boldsymbol{n},\mathsf{U}-\hat{u}_h\rangle_{\partial K} = (f,\mathsf{U})_K - \langle\hat{\mathbf{Q}}\cdot\boldsymbol{n},\hat{u}_h\rangle_{\partial K}.$$

The energy identity now follows by simply inserting the definition of the numerical trace $\hat{\mathbf{Q}}$. This completes the proof. □

We are now ready to present our main result. It is a variation of a similar result in [33].

Theorem 4 *Assume that the stabilization function τ satisfies the nonnegativity condition (2). Assume also that, for each element $K \in \Omega_h$, we have that if $(w, \mu) \in W(K) \times M(\partial K)$ is such that*

$$\text{(i)} \quad \langle \tau(w-\mu), w-\mu \rangle_{\partial K} = 0,$$

$$\text{(ii)} \quad (\nabla w, \boldsymbol{v})_K + \langle \mu - w, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0 \ \forall \ \boldsymbol{v} \in \boldsymbol{V}(K),$$

then w is a constant on K and $w = \mu$ on ∂K. Then the approximate solution $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ of the HDG method is well defined.

Note that condition (ii) establishes a relation between the local spaces $\boldsymbol{V}(K)$, $W(K)$ and $M(\partial K)$ and the stabilization function τ *guaranteeing* that the local problems as well as the global problem have a unique solution. Note also that if condition (i) were not necessary to obtain that w is a constant on K and $w = \mu$ on ∂K, we can simply take $\tau \equiv 0$. However, for most cases, without condition (i), the method might simply fail to be well defined. The role of τ, is thus to *prevent* this failure.

Let us now prove Theorem 4.

Proof Since the HDG method defines a finite dimensional square system for the unknowns $(\mathbf{Q}, \mathsf{U}, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h$, we only have to show that, when we set the data f and u_D to zero, the only solution is the trivial one.

Thus, setting $\mu := \hat{u}_h$ in the transmission condition, and recalling that, by the boundary condition, $\hat{u}_h = 0$ on $\partial\Omega$, we get

$$0 = -\langle \hat{u}_h, \hat{\mathbf{Q}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} = (\mathsf{c}\,\mathbf{Q}, \mathbf{Q})_{\Omega_h} + \langle (\mathsf{U} - \hat{u}_h), \tau(\mathsf{U} - \hat{u}_h) \rangle_{\partial\Omega_h},$$

by the energy identity of the previous proposition. By assumption (i), we get that $\mathbf{Q} = 0$ on Ω and that $\langle (\mathsf{U} - \hat{u}_h), \tau(\mathsf{U} - \hat{u}_h) \rangle_{\partial K} = 0$ for any $K \in \Omega_h$. Moreover, the first equation defining the local problem now reads

$$(\nabla \mathsf{U}, \boldsymbol{v})_K + \langle \hat{u}_h - \mathsf{U}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0 \ \forall \ \boldsymbol{v} \in \boldsymbol{V}(K).$$

By assumption (ii) with $(w, \mu) := (\mathsf{U}, \hat{u}_h)$, we have that, on each element $K \in \Omega_h$, U is a constant on K and that $\mathsf{U} = \hat{u}_h$ on ∂K. As a consequence, U is a constant on Ω and $\mathsf{U} = \hat{u}_h$ on $\mathcal{F}_h$. Since $\hat{u}_h = 0$ on $\partial\Omega$ we finally get that $\mathsf{U} = 0$ on Ω and that $\hat{u}_h = 0$ on $\mathcal{F}_h$. This completes the proof. □

Let us now present an almost direct consequence of the previous result in a case in which the stabilization function τ is very *strong*.

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Corollary 1 ([33]) *Assume that the stabilization function* τ *satisfies the nonnegativity condition (2). Assume also that, for every element* $K \in \Omega_h$,

(a) $(w, \mu) \in W(K) \times M(\partial K) : \langle \tau(w - \mu), w - \mu \rangle_{\partial K} = 0 \implies w = \mu$ *on* ∂K,
(b) $\nabla W(K) \subset \boldsymbol{V}(K)$.

Then the approximate solution $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h$ *of the HDG method is well defined.*

A remarkable feature of this result is that the method is well defined *completely independently* of the choice of the space M_h. This is a direct consequence of condition (a), which is clearly stronger than condition (i) of Theorem 4 on the stabilization function τ. Thanks to condition (a) , we can replace condition (ii) of Theorem 4 by the simpler condition (b), as we see next.

Proof We only have to show that the assumptions the previous result are satisfied. Since τ is a linear mapping, assumption (a) implies condition (i) of Theorem 4. Now, by assumption (a), if $\langle \tau(w - \mu), w - \mu \rangle_{\partial K} = 0$, we have that $w = \mu$ on ∂K and we get that condition (ii) of Theorem 4 reads

$$(\nabla w, \boldsymbol{v})_K = 0 \ \forall \boldsymbol{v} \in \boldsymbol{V}(K).$$

By assumption (b), we can take $\boldsymbol{v} := \nabla w$ and conclude that w is a constant on K. This implies that the second assumption of Theorem 4 holds. This completes the proof. □

3.3 Characterizations of the HDG Methods

Here, we provide two characterizations of the approximate solution provided by the HDG methods just introduced. We are going to use the set

$$M_h(g) := \{\mu \in M_h : \langle \mu, \eta \rangle_{\partial\Omega} := \langle g, \eta \rangle_{\partial\Omega} \forall \eta \in M_h\}.$$

3.3.1 Formulation in Terms of $(\boldsymbol{q}_h, u_h, \hat{u}_h)$

Static Condensation Formulation

The following result reflects the way in which the HDG methods were devised and renders evident the way in which their *implementation* by static condensation can be achieved.

Theorem 5 (First Characterization of HDG Methods) *The approximate solution of the HDG method is given by*

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{Q}, \mathsf{U}) = (\boldsymbol{Q}_{\hat{u}_h}, \mathsf{U}_{\hat{u}_h}) + (\boldsymbol{Q}_f, \mathsf{U}_f),$$

where, on the element $K \in \Omega_h$*, for any* $\mu \in L^2(\partial K)$*, the function* $(\boldsymbol{Q}_\mu, \mathsf{U}_\mu) \in \boldsymbol{V}(K) \times W(K)$ *is the solution of the local problem*

$$\begin{aligned}(\mathsf{c}\,\boldsymbol{Q}_\mu, \boldsymbol{v})_K - (\mathsf{U}_\mu, \nabla\cdot\boldsymbol{v})_K + \langle \mu, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\ -(\boldsymbol{Q}_\mu, \nabla w)_K + \langle \hat{\boldsymbol{Q}}_\mu\cdot\boldsymbol{n}, w\rangle_{\partial K} &= 0 \quad \forall w \in W(K),\\ \hat{\boldsymbol{Q}}_\mu\cdot\boldsymbol{n} := \boldsymbol{Q}_\mu\cdot\boldsymbol{n} + \tau(\mathsf{U}_\mu - \mu) & \qquad \textit{on } \partial K,\end{aligned}$$

and, for any $f \in L^2(K)$*, the function* $(\boldsymbol{Q}_f, \mathsf{U}_f) \in \boldsymbol{V}(K) \times W(K)$ *is the solution of the local problem*

$$\begin{aligned}(\mathsf{c}\,\boldsymbol{Q}_f, \boldsymbol{v})_K - (\mathsf{U}_f, \nabla\cdot\boldsymbol{v})_K &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\ -(\boldsymbol{Q}_f, \nabla w)_K + \langle \hat{\boldsymbol{Q}}_f\cdot\boldsymbol{n}, w\rangle_{\partial K} &= (f, w)_K && \forall w \in W(K),\\ \hat{\boldsymbol{Q}}_f\cdot\boldsymbol{n} := \boldsymbol{Q}_f\cdot\boldsymbol{n} + \tau(\mathsf{U}_f) & && \textit{on } \partial K.\end{aligned}$$

The function $\hat{u}_h$ *is the element of* $M_h(u_D)$ *such that*

$$a_h(\hat{u}_h, \mu) = \ell_h(\mu) \quad \forall\ \mu \in M_h(0),$$

where $a_h(\mu, \lambda) := -\langle \mu, \hat{\boldsymbol{Q}}_\lambda\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}$*, and* $\ell_h(\mu) := \langle \mu, \hat{\boldsymbol{Q}}_f\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}$*. Moreover,*

$$a_h(\mu, \lambda) = (\mathsf{c}\,\boldsymbol{Q}_\mu, \boldsymbol{Q}_\lambda)_{\Omega_h} + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}, \quad \ell_h(\mu) = (f, \mathsf{U}_\mu),$$

and $a_h(\cdot,\cdot)$ *is symmetric and positive definite on* $M_h(0) \times M_h(0)$*. Thus,* $\hat{u}_h$ *minimizes the total energy functional* $J_h(\mu) := \frac{1}{2}a_h(\mu,\mu) - \ell_h(\mu)$ *over* $M_h(u_D)$*.*

Proof We only need to prove the last two identities and the property of positive definiteness of the bilinear form $a_h(\cdot,\cdot)$.

Let us prove the first identity. If we take $\boldsymbol{v} := \mathbf{Q}_\lambda$ in the first equation defining the first local problem, replace μ by λ in the second equation defining the first local problem and set $w := \mathsf{U}_\mu$, we get

$$\begin{aligned}(\mathsf{c}\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K - (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_\lambda)_K + \langle \mu, \mathbf{Q}_\lambda\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{Q}_\lambda, \nabla \mathsf{U}_\mu)_K + \langle \hat{\mathbf{Q}}_\lambda\cdot\boldsymbol{n}, \mathsf{U}_\mu\rangle_{\partial K} &= 0.\end{aligned}$$

Adding the two equations, we obtain

$$(\mathsf{c}\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K + \langle (\hat{\mathbf{Q}}_\lambda - \mathbf{Q}_\lambda)\cdot \boldsymbol{n}, \mathsf{U}_\mu - \mu\rangle_{\partial K} = -\langle \hat{\mathbf{Q}}_\lambda \cdot \boldsymbol{n}, \mu\rangle_{\partial K}.$$

The first identity follows by inserting the definition of the numerical trace $\hat{\mathbf{Q}}_\lambda$ and adding over the elements $K \in \Omega_h$.

Let us prove the second identity. If we take $\boldsymbol{v} := \mathbf{Q}_f$ in the first equation defining the first local problem and $w := \mathsf{U}_\mu$ in the second equation defining the second local problem, we get

$$\begin{aligned}(\mathsf{c}\,\mathbf{Q}_\mu, \mathbf{Q}_f)_K - (\mathsf{U}_\mu, \nabla\cdot\mathbf{Q}_f)_K + \langle \mu, \mathbf{Q}_f\cdot\boldsymbol{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{Q}_f, \nabla \mathsf{U}_\mu)_K + \langle \hat{\mathbf{Q}}_f\cdot\boldsymbol{n}, \mathsf{U}_\mu\rangle_{\partial K} &= (f, \mathsf{U}_\mu)_k\end{aligned}$$

and if we add the two equations and insert the definition of $\hat{\mathbf{Q}}_f$, we obtain

$$(\mathsf{c}\,\mathbf{Q}_\mu, \mathbf{Q}_f)_K + \langle \tau(\mathsf{U}_f), \mathsf{U}_\mu - \mu\rangle_{\partial K} = (f, \mathsf{U}_\mu)_K - \langle \hat{\mathbf{Q}}_f\cdot\boldsymbol{n}, \mu\rangle_{\partial K}.$$

If we now take $\boldsymbol{v} := \mathbf{Q}_\mu$ in the first equation defining the second local problem and $w := \mathsf{U}_f$ in the second equation defining the first local problem with $\hat{u}_h := \mu$, we get

$$\begin{aligned}(\mathsf{c}\,\mathbf{Q}_f, \mathbf{Q}_\mu)_K - (\mathsf{U}_f, \nabla\cdot\mathbf{Q}_\mu)_K &= 0,\\ -(\mathbf{Q}_\mu, \nabla \mathsf{U}_f)_K + \langle \hat{\mathbf{Q}}_\mu\cdot\boldsymbol{n}, \mathsf{U}_f\rangle_{\partial K} &= 0,\end{aligned}$$

and if we proceed as before, we get

$$(\mathsf{c}\,\mathbf{Q}_f, \mathbf{Q}_\mu)_K + \langle \tau(\mathsf{U}_\mu - \mu), \mathsf{U}_f\rangle_{\partial K} = 0.$$

This implies that

$$-\langle \tau(\mathsf{U}_\mu - \mu), \mathsf{U}_f\rangle_{\partial K} + \langle \tau(\mathsf{U}_f), \mathsf{U}_\mu - \mu\rangle_{\partial K} = (f, \mathsf{U}_\mu)_K - \langle \hat{\mathbf{Q}}_f\cdot\boldsymbol{n}, \mu\rangle_{\partial K},$$

and the result follows by the fact that τ is symmetric.

The fact that $a_h(\cdot,\cdot)$ is symmetric follows from the previous identities and the fact that τ is also symmetric. Finally the fact that it is positive definite on $M_h(0)\times M_h(0)$ follows exactly as in the proof of Theorem 4. This completes the proof. □

Two Compact Formulations

Let us now show how to rewrite the HDG methods in a more *compact* manner. It does not suggest a way to statically condense the methods but it is our *favorite* way of presenting them concisely. It emphasizes the role of the numerical traces of the methods and is suitable for carrying out their analysis. It is the following.

The approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$(\mathrm{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0 \qquad \forall \boldsymbol{v}\in\boldsymbol{V}_h, \tag{3a}$$

$$-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h} \quad \forall w\in W_h, \tag{3b}$$

$$\hat{\boldsymbol{q}}_h\cdot\boldsymbol{n} := \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau(u_h - \hat{u}_h) \qquad \text{on } \partial\Omega_h, \tag{3c}$$

$$\langle \mu, \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0 \qquad \forall \mu\in M_h(0). \tag{3d}$$

Indeed, note that the first, second and third equations correspond to the definition of the local problems and that the weakly imposed boundary conditions are enforced by requesting that $\hat{u}_h$ be an element of $M_h(u_D)$.

We can also eliminate the numerical trace $\hat{\boldsymbol{q}}_h$ to obtain yet another rewriting of the methods. Once again, it hides the numerical trace of the flux, but emphasizes what we could call the *stabilized mixed method* structure of the methods. The formulation is the following. The approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned} A_h(\boldsymbol{q}_h, \boldsymbol{v}) + B_h(u_h, \hat{u}_h; \boldsymbol{v}) &= 0 && \forall \boldsymbol{v}\in\boldsymbol{V}_h,\\ -B_h(w,\mu;\boldsymbol{q}_h) + S_h(u_h,\hat{u}_h; w,\mu) &= (f,w)_{\Omega_h} && \forall (w,\mu)\in W_h\times M_h(0), \end{aligned}$$

where

$$\begin{aligned} A_h(\boldsymbol{p}, \boldsymbol{v}) &:= (\mathrm{c}\,\boldsymbol{p}, \boldsymbol{v})_{\Omega_h} && \forall \boldsymbol{p}, \boldsymbol{v}\in\boldsymbol{V}_h,\\ B_h(w,\mu;\boldsymbol{v}) &:= -(w, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle\mu, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} && \forall (\boldsymbol{v}, w, \mu)\in\boldsymbol{V}_h\times W_h\times M_h,\\ S_h(\omega,\lambda; w,\mu) &:= \langle\tau(\omega-\lambda), w-\mu\rangle_{\partial\Omega_h} && \forall (\omega,\lambda), (w,\mu)\in W_h\times M_h. \end{aligned}$$

Indeed, the first equation follows from the definition of the bilinear forms $A_h(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$. It remains to prove that

$$B_h(w,\mu;\boldsymbol{q}_h) + S_h(u_h,\hat{u}_h; w,\mu) = -(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} - \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h}.$$

But, we have, by the definition of the bilinear forms $B_h(\cdot,\cdot)$ and $S_h(\cdot,\cdot)$, that

$$\begin{aligned} \Theta :=& - B_h(w,\mu;\boldsymbol{q}_h) + S_h(u_h,\hat{u}_h; w,\mu)\\ =& \,(w, \nabla\cdot\boldsymbol{q})_{\Omega_h} - \langle\mu, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle\tau(u_h-\hat{u}_h), w-\mu\rangle_{\partial\Omega_h}\\ =& \,-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \boldsymbol{q}\cdot n + \tau(u_h-\hat{u}_h), w-\mu\rangle_{\partial\Omega_h}, \end{aligned}$$

by integration by parts. The identity we want follows now by using the definition of the numerical trace of the flux.

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To end, we note that, thanks to the structure of the methods, it is easy to see that the solution $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h(u_D)$ minimizes the functional

$$J_h(\boldsymbol{v}, w, \mu) := \frac{1}{2}\{A_h(\boldsymbol{v}, \boldsymbol{v}) + S_h(w, \mu; w, \mu)\} - (f, w)_{\Omega_h} \tag{4a}$$

over all functions $(\boldsymbol{v}, w, \mu)$ in $\boldsymbol{V}_h \times W_h \times M_h(u_D)$ such that

$$A_h(\boldsymbol{v}, \boldsymbol{p}) + B_h(w, \mu; \boldsymbol{p}) = 0 \ \forall \boldsymbol{p} \in \boldsymbol{V}_h. \tag{4b}$$

Note that the last equation can be interpreted as the *elimination* of $\boldsymbol{q}_h$ from the equations. The minimization problem would then be one on the affine space $W_h \times M_h(u_D)$ and would correspond to a problem formulated solely in terms of u_h and $\hat{u}_h$. Next, we explore the static condensation of such reformulation.

3.3.2 Formulation in Terms of $(u_h, \hat{u}_h)$

So, here we *eliminate* the approximate flux $\boldsymbol{q}_h$ from the equations defining the HDG method in order to formulate it solely in terms of $(u_h, \hat{u}_h)$. To achieve this, we simply rewrite $\boldsymbol{q}_h$ as a linear mapping applied to $(u_h, \hat{u}_h)$. This mapping is defined by using the first equation defining the HDG methods. Thus, for any $(w, \mu) \in W_h \times M_h$, we define $\mathbf{Q}_{w,\mu} \in \boldsymbol{V}_h$ as the solution of

$$(\mathsf{c}\,\mathbf{Q}_{w,\mu}, \boldsymbol{v})_{\Omega_h} - (w, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \mu, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \Omega_h} = 0 \quad \forall \ \boldsymbol{v} \in \boldsymbol{V}_h,$$

In this way, we are going to have that $\boldsymbol{q}_h = \mathbf{Q}_{u_h,\hat{u}_h}$. Note that the above equation is nothing but a *rewriting* of Eq. (4b).

Static Condensation Formulation

Using this approach, we obtain the following characterization of the HDG methods. It is useful for their implementation and involves less unknowns than the previous characterization since the unknown for the approximate flux has been eliminated. (Of course, the price to pay for this is that we now we have to work with the mapping $(w, \mu) \mapsto \mathbf{Q}_{w,\mu}$.) This characterization better shows the role of τ as a stabilization function but it hides its relation with the numerical trace of the flux and does not clearly indicate the associated transmission condition.

Theorem 6 (Second Characterization of HDG Methods) *The approximate solution of the HDG method is given by*

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{Q}, \mathsf{U}) = (\boldsymbol{Q}_{\mathsf{U}_{\hat{u}_h},\hat{u}_h}, \mathsf{U}_{\hat{u}_h}) + (\boldsymbol{Q}_{\mathsf{U}_f,0}, \mathsf{U}_f),$$

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where, on the element $K \in \Omega_h$, for any $\mu \in L^2(\partial K)$ and any $f \in L^2(K)$, the functions $\mathsf{U}_\mu, \mathsf{U}_f \in W(K)$ are the solutions of the local problems

$$(\mathsf{c}\,\boldsymbol{Q}_{\mathsf{U}_\mu,\mu}, \boldsymbol{Q}_{w,0})_K + \langle \tau(\mathsf{U}_\mu - \mu), w\rangle_{\partial K} = 0 \qquad \forall w \in W(K),$$
$$(\mathsf{c}\,\boldsymbol{Q}_{\mathsf{U}_f,0}, \boldsymbol{Q}_{w,0})_K + \langle \tau(\mathsf{U}_f), w\rangle_{\partial K} = (f, w)_K \quad \forall w \in W(K),$$

respectively. The function $\hat{u}_h$ is the element of $M_h(u_D)$ such that

$$a_h(\hat{u}_h, \mu) = \ell_h(\mu) \quad \forall\ \mu\ \in\ M_h(0),$$

where $a_h(\mu, \lambda) := -\langle \mu, \hat{\boldsymbol{Q}}_{\mathsf{U}_\lambda,\lambda} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}$ and $\ell_h(\mu) := \langle \mu, \hat{\boldsymbol{Q}}_{\mathsf{U}_f,0} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}$. Moreover,

$$a_h(\mu, \lambda) = (\mathsf{c}\ \boldsymbol{Q}_{\mathsf{U}_\mu,\mu}, \boldsymbol{Q}_{\mathsf{U}_\lambda,\lambda})_{\partial\Omega_h} + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}, \quad \ell_h(\mu) = (f, \mathsf{U}_\mu),$$

and $a_h(\cdot,\cdot)$ is symmetric and positive definite on $M_h(0) \times M_h(0)$. Thus, $\hat{u}_h$ minimizes the total energy functional $J_h(\mu) := \frac{1}{2} a_h(\mu, \mu) - \ell_h(\mu)$ over $M_h(u_D)$.

Proof This results follows easily from the first characterization of the HDG methods given in Theorem 5. We only have to show that the solutions of the local problems coincide, that is, that $(\mathbf{Q}_{\mathsf{U}_\mu,\mu}, \mathsf{U}_\mu) \in \boldsymbol{V}(K) \times W(K)$ is the solution of

$$(\mathsf{c}\,\mathbf{Q}_{\mathsf{U}_\mu,\mu}, \boldsymbol{v})_K - (\mathsf{U}_\mu, \nabla\cdot \boldsymbol{v})_K + \langle \mu, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}(K),$$
$$-(\mathbf{Q}_{\mathsf{U}_\mu,\mu}, \nabla w)_K + \langle \hat{\mathbf{Q}}_{\mathsf{U}_\mu,\mu} \cdot \boldsymbol{n}, w\rangle_{\partial K} = 0 \quad \forall w \in W(K),$$
$$\hat{\mathbf{Q}}_\mu \cdot \boldsymbol{n} := \mathbf{Q}_\mu \cdot \boldsymbol{n} + \tau(\mathsf{U}_\mu - \mu) \qquad \text{on } \partial K,$$

and that $(\mathbf{Q}_{\mathsf{U}_f,0}, \mathsf{U}_f) \in \boldsymbol{V}(K) \times W(K)$ is the solution of

$$(\mathsf{c}\,\mathbf{Q}_{\mathsf{U}_f,0}, \boldsymbol{v})_K - (\mathsf{U}_f, \nabla\cdot \boldsymbol{v})_K = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}(K),$$
$$-(\mathbf{Q}_{\mathsf{U}_f,0}, \nabla w)_K + \langle \hat{\mathbf{Q}}_{\mathsf{U}_f,0} \cdot \boldsymbol{n}, w\rangle_{\partial K} = (f, w)_K \quad \forall w \in W(K),$$
$$\hat{\mathbf{Q}}_f \cdot \boldsymbol{n} := \mathbf{Q}_f \cdot \boldsymbol{n} + \tau(\mathsf{U}_f) \qquad \text{on } \partial K.$$

Since the first equation of these problems is nothing but the definition of the operator $\mathbf{Q}_{w,\mu}$, we only have to show that

$$(\nabla\cdot \mathbf{Q}_{\mathsf{U}_\mu,\mu}, w)_K + \langle \tau(\mathsf{U}_\mu - \mu), w\rangle_{\partial K} = 0 \qquad \forall w \in W(K),$$
$$(\nabla\cdot \mathbf{Q}_{\mathsf{U}_f,0}, w)_K + \langle \tau(\mathsf{U}_f), w\rangle_{\partial K} = (f, w)_K \quad \forall w \in W(K).$$

But, by the definition of $\mathbf{Q}_{w,0}$, we have

$$(\nabla \cdot \mathbf{Q}_{\mathsf{U}_\mu,\mu}, w)_K = (\mathsf{c}\,\mathbf{Q}_{w,0}, \mathbf{Q}_{\mathsf{U}_\mu,\mu})_K,$$
$$(\nabla \cdot \mathbf{Q}_{\mathsf{U}_f,0}, w)_K = (\mathsf{c}\,\mathbf{Q}_{w,0}, \mathbf{Q}_{\mathsf{U}_f,0})_K,$$

and the result follows. This completes the proof. □

A Compact Formulation

As we did for the first characterization of the HDG methods, we can rewrite the above result in a *compact* manner as follows. The approximate flux provided by the HDG method is $\boldsymbol{q}_h = \mathbf{Q}_{u_h,\hat{u}_h}$ and $(u_h, \hat{u}_h) \in W_h \times M_h(u_D)$ is the solution of

$$(\mathsf{c}\,\mathbf{Q}_{u_h,\hat{u}_h}, \mathbf{Q}_{w,\mu})_{\Omega_h} + \langle \tau(u_h - \hat{u}_h), w - \mu\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h} \quad \forall (w, \mu) \in W_h \times M_h(0).$$

We immediately see that $(u_h, \hat{u}_h)$ is the only minimum over $W_h \times M_h(0)$ of the *total energy* functional

$$J_h(w, \mu) := \frac{1}{2}\{(\mathsf{c}\,\mathbf{Q}_{w,\mu}, \mathbf{Q}_{w,\mu})_{\Omega_h} + \langle \tau(w - \mu), w - \mu\rangle_{\partial\Omega_h}\} - (f, w)_{\Omega_h}.$$

This minimization problem is *identical* to the minimization (with restrictions) problem (4).

4 HDG Methods Using Only the Tensor $\mathsf{a} := \mathsf{c}^{-1}$

4.1 Motivation

Note that the the first three equations of the weak formulation of the DG methods we have been considering can also be expressed as

$$-(\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = 0,$$
$$(\mathsf{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} = -(\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h},$$
$$-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h},$$

where the approximate gradient $\boldsymbol{g}_h$ is taken in $\boldsymbol{V}_h$ If one prefers to work with the tensor $\mathsf{a} := \mathsf{c}^{-1}$, we can simply use the equations

$$-(\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = 0,$$
$$(\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} = -(\mathsf{a}\,\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h},$$
$$-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h},$$

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for all $(\boldsymbol{v}, w) \in \boldsymbol{V}_h \times W_h$, where the numerical traces $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}$ are approximations to $u|_{\partial\Omega_h}$ and $\boldsymbol{q} \cdot \boldsymbol{n}|_{\partial\Omega_h}$, respectively. The difference between these two DG methods is certainly not abysmal since it consists in picking one of the two ways of relating the approximate gradient $\boldsymbol{g}_h$ to the approximate flux $\boldsymbol{q}_h$, namely,

$$(\mathsf{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} = -(\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h} \quad \text{or} \quad (\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} = -(\mathsf{a}\,\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h}.$$

As a consequence, there is a one-to-one correspondence between these two weak formulations, provided both a and c are well defined. Moreover, both formulations coincide whenever a and c are constant on each element $K \in \Omega_h$ which gives rise to super-closeness of their approximations, as noted in [45].

However, if a degenerates and is not invertible at every point, the second formulation might be preferable. This is also what motivated the so-called "extended" form of the mixed methods introduced in [1, 9, 61].

Finally, let us note that in elasticity, $\boldsymbol{g}$ corresponds to the strain, $\boldsymbol{q}$ to the stress, a to the so-called constitutive tensor and c to the so-called compliance tensor. Thus, the HDG methods obtained for linear and nonlinear elasticity, see the HDG methods for elasticity considered in 2008 [85], 2009 [86] and 2014 [53] and in 2015 [59], can be immediately *reduced* to our simpler case; see also the 2006 DG method proposed in [87]. It is well known that to work with the constitutive tensor is usually preferred in the case of nonlinear elasticity. Next, we briefly show how to define and characterize the HDG methods associated with using the tensor $\mathsf{a} := \mathsf{c}^{-1}$.

4.2 *Definition, Existence and Uniqueness*

We take the approximate solution of the HDG methods to be the function

$$(\boldsymbol{q}_h, \boldsymbol{g}_h, u_h) = (\mathbf{Q}, \mathbf{G}, \mathsf{U}),$$

where, on the element $K \in \Omega_h$, $(\mathbf{Q}, \mathbf{G}, \mathsf{U}) \in \boldsymbol{V}(K) \times \boldsymbol{V}(K) \times W(K)$ is the solution of the local problem

$$\begin{aligned} -(\mathbf{G}, \boldsymbol{v})_K - (\mathsf{U}, \nabla \cdot \boldsymbol{v})_K + \langle \hat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K), \\ (\mathbf{Q}, \boldsymbol{v})_K &= -(\mathsf{a}\,\mathbf{G}, \boldsymbol{v})_K && \forall \boldsymbol{v} \in \boldsymbol{V}(K), \\ -(\mathbf{Q}, \nabla w)_K + \langle \hat{\mathbf{Q}} \cdot \boldsymbol{n}, w \rangle_{\partial K} &= (f, w)_K && \forall w \in W(K), \end{aligned}$$

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where the numerical trace $\hat{\mathbf{Q}}$ is suitably chosen, and $\hat{u}_h \in M_h$ is the solution of the following weakly imposed transmission and boundary conditions:

$$\langle \mu, \hat{\mathbf{Q}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h \setminus \partial\Omega} = 0,$$
$$\langle \mu, \hat{u}_h \rangle_{\partial\Omega} = \langle \mu, u_D \rangle_{\partial\Omega},$$

for all $\mu \in M_h$. This completes the definition of the HDG methods.

It is not difficult to see that the existence and uniqueness in Theorem 4 and its Corollary 1 do hold *unchanged*.

4.3 Characterizations of the HDG Methods

4.3.1 Formulation in Terms of $(\boldsymbol{q}_h, \boldsymbol{g}_h, u_h, \hat{u}_h)$

Static Condensation Formulation

We have the following result which is analogous to Theorem 5.

Theorem 7 (First Characterization of HDG Methods) *The approximate solution of the HDG method is given by*

$$(\boldsymbol{q}_h, \boldsymbol{g}_h, u_h) = (\boldsymbol{Q}, \boldsymbol{G}, \mathsf{U}) = (\boldsymbol{Q}_{\hat{u}_h}, \boldsymbol{G}_{\hat{u}_h}, \mathsf{U}_{\hat{u}_h}) + (\boldsymbol{Q}_f, \boldsymbol{G}_f, \mathsf{U}_f),$$

where, on the element $K \in \Omega_h$, for any $\mu \in L^2(\partial K)$, the function $(\boldsymbol{Q}_\mu, \boldsymbol{G}_\mu, \mathsf{U}_\mu) \in \boldsymbol{V}(K) \times \boldsymbol{V}(K) \times W(K)$ is the solution of the local problem

$$\begin{aligned}
-(\boldsymbol{G}_\mu, \boldsymbol{v})_K - (\mathsf{U}_\mu, \nabla \cdot \boldsymbol{v})_K + \langle \mu, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
(\boldsymbol{Q}_\mu, \boldsymbol{v})_K &= -(\mathsf{a}\boldsymbol{G}_\mu, \boldsymbol{v})_K && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
-(\boldsymbol{Q}_\mu, \nabla w)_K + \langle \hat{\boldsymbol{Q}}_\mu \cdot \boldsymbol{n}, w \rangle_{\partial K} &= 0 && \forall w \in W(K),\\
\hat{\boldsymbol{Q}}_\mu \cdot \boldsymbol{n} := \boldsymbol{Q}_\mu \cdot \boldsymbol{n} + \tau(\mathsf{U}_\mu - \mu) & && \textit{on } \partial K,
\end{aligned}$$

and, for any $f \in L^2(K)$, the function $(\boldsymbol{Q}_f, \boldsymbol{G}_f, \mathsf{U}_f) \in \boldsymbol{V}(K) \times \boldsymbol{V}(K) \times W(K)$ is the solution of the local problem

$$\begin{aligned}
-(\boldsymbol{G}_f, \boldsymbol{v})_K - (\mathsf{U}_f, \nabla \cdot \boldsymbol{v})_K &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
(\boldsymbol{Q}_f, \boldsymbol{v})_K &= -(\mathsf{a}\boldsymbol{G}_f, \boldsymbol{v})_K && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
-(\boldsymbol{Q}_f, \nabla w)_K + \langle \hat{\boldsymbol{Q}}_f \cdot \boldsymbol{n}, w \rangle_{\partial K} &= (f, w)_K && \forall w \in W(K),\\
\hat{\boldsymbol{Q}}_f \cdot \boldsymbol{n} := \boldsymbol{Q}_f \cdot \boldsymbol{n} + \tau(\mathsf{U}_f) & && \textit{on } \partial K.
\end{aligned}$$

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The function $\hat{u}_h$ is the element of $M_h(u_D)$ such that

$$a_h(\hat{u}_h, \mu) = \ell_h(\mu) \quad \forall\ \mu\ \in\ M_h(0),$$

where $a_h(\mu, \lambda) := -\langle \mu, \hat{\boldsymbol{Q}}_\lambda \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}$, and $\ell_h(\mu) := \langle \mu, \hat{\boldsymbol{Q}}_f \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}$. Moreover,

$$a_h(\mu, \lambda) = (\mathsf{a}\boldsymbol{G}_\mu, \boldsymbol{G}_\lambda)_{\partial\Omega_h} + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda) \rangle_{\partial\Omega_h}, \quad \ell_h(\mu) = (f, \mathsf{U}_\mu),$$

and $a_h(\cdot,\cdot)$ is symmetric and positive definite on $M_h(0) \times M_h(0)$. Thus, $\hat{u}_h$ minimizes the functional $J_h(\mu) := \frac{1}{2} a_h(\mu, \mu) - \ell_h(\mu)$ over $M_h(u_D)$.

Two Compact Formulations

Proceeding as for the first family of HDG methods, we obtain the following two compact formulations. The first emphasized the role of the numerical traces. It reads as follows. The approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, \boldsymbol{g}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned}
-(\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
(\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} &= -(\mathsf{a}\boldsymbol{g}_h, \boldsymbol{v})_{\Omega_h} && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h} && \forall w \in W_h,\\
\hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} := \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau(u_h - \hat{u}_h) & && \text{on } \partial\Omega_h,\\
\langle \mu, \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} &= 0 && \forall \mu \in M_h(0).
\end{aligned}$$

The second emphasizes the stabilized mixed structure of the method. It is the following. The approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$\begin{aligned}
A_h(\boldsymbol{g}_h, \boldsymbol{v}) + \mathsf{B}_h(\boldsymbol{q}_h, \boldsymbol{v}) &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
-\mathsf{B}_h(\boldsymbol{v}, \boldsymbol{g}_h) + B_h(u_h, \hat{u}_h; \boldsymbol{v}) &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
-B_h(w, \mu; \boldsymbol{q}_h) + S_h(u_h, \hat{u}_h; w, \mu) &= (f, w)_{\Omega_h} && \forall (w, \mu) \in W_h \times M_h(0),
\end{aligned}$$

where

$$\begin{aligned}
A_h(\boldsymbol{p}, \boldsymbol{v}) &:= (\mathsf{a}\boldsymbol{p}, \boldsymbol{v})_{\Omega_h}, && \forall \boldsymbol{p}, \boldsymbol{v} \in \boldsymbol{V}_h,\\
\mathsf{B}_h(\boldsymbol{p}, \boldsymbol{v}) &:= (\boldsymbol{p}, \boldsymbol{v})_{\Omega_h}, && \forall \boldsymbol{p}, \boldsymbol{v} \in \boldsymbol{V}_h,\\
B_h(w, \mu; \boldsymbol{v}) &:= -(w, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \mu, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} && \forall (\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h \times W_h \times M_h,\\
S_h(\omega, \lambda; w, \mu) &:= \langle \tau(\omega - \lambda), w - \mu \rangle_{\partial\Omega_h} && \forall (\omega, \lambda), (w, \mu) \in W_h \times M_h.
\end{aligned}$$

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Thanks to the structure of the method, it is easy to see that the solution $(\boldsymbol{g}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h(u_D)$ minimizes the functional

$$J_h(\boldsymbol{v}, w, \mu) := \frac{1}{2}\{A_h(\boldsymbol{v}, \boldsymbol{v}) + S_h(w, \mu; w, \mu)\} - (f, w)_{\Omega_h} \tag{5a}$$

over the functions $(\boldsymbol{v}, w, \mu)$ in the space $\boldsymbol{V}_h \times W_h \times M_h(u_D)$ such that there exist $\boldsymbol{q}_h = \boldsymbol{q}_h(\boldsymbol{v}, w, \mu) \in \boldsymbol{V}_h$ such that

$$A_h(\boldsymbol{v}, \boldsymbol{p}) + \mathsf{B}_h(\boldsymbol{q}_h, \boldsymbol{p}) = 0 \quad \forall \boldsymbol{p} \in \boldsymbol{V}_h, \tag{5b}$$

$$-\mathsf{B}_h(\boldsymbol{p}, \boldsymbol{v}) + B_h(w, \mu; \boldsymbol{p}) = 0 \quad \forall \boldsymbol{p} \in \boldsymbol{V}_h. \tag{5c}$$

Once again, Note that the last two equations can be interpreted as the *elimination* of $(\boldsymbol{q}_h, \boldsymbol{g}_h)$ from the equations. The minimization problem would then be one on the affine space $W_h \times M_h(u_D)$ and would correspond to a problem formulated solely in terms of u_h and $\hat{u}_h$. Next, we explore such reformulation.

4.3.2 Formulation in Terms of $(u_h, \hat{u}_h)$

We *eliminate* the approximate gradient $\boldsymbol{g}_h$ and the approximate flux $\boldsymbol{q}_h$ from the equations defining the HDG method in order to formulate it solely in terms of $(u_h, \hat{u}_h)$. To achieve that, we simply rewrite $\boldsymbol{g}_h$ and $\boldsymbol{q}_h$ as a linear mappings applied to $(u_h, \hat{u}_h)$. These mappings are defined by using the first equation defining the HDG methods. Thus, for any $(w, \mu) \in W_h \times M_h$, we define $(\mathbf{G}_{w,\mu}, \mathbf{Q}_{w,\mu}) \in \boldsymbol{V}_h \times \boldsymbol{V}_h$ as the solution of

$$-(\mathbf{G}_{w,\mu}, \boldsymbol{v})_{\Omega_h} - (w, \nabla \cdot \boldsymbol{v})_{\Omega_h} + \langle \mu, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} = 0 \qquad \forall\ \boldsymbol{v} \in \boldsymbol{V}_h,$$
$$(\mathbf{Q}_{w,\mu}, \boldsymbol{v})_{\Omega_h} = -(\mathsf{a}\, \mathbf{G}_{w,\mu}, \boldsymbol{v})_{\Omega_h} \qquad \forall\ \boldsymbol{v} \in \boldsymbol{V}_h.$$

In this way, we are going to have that $(\boldsymbol{q}_h, \boldsymbol{g}_h) = (\mathbf{Q}_{u_h,\hat{u}_h}, \mathbf{G}_{u_h,\hat{u}_h})$. Note that these two equations are nothing but a rewriting of Eqs. (5b) and (5c).

Static Condensation Formulation

We have the following result.

Theorem 8 (Second Characterization of HDG Methods) *The approximate solution of the HDG method is given by*

$$(\boldsymbol{q}_h, \boldsymbol{g}_h, u_h) = (\boldsymbol{Q}, \boldsymbol{G}, \mathsf{U}) = (\boldsymbol{Q}_{\mathsf{U}_{\hat{u}_h}, \hat{u}}, \boldsymbol{G}_{\mathsf{U}_{\hat{u}_h}, \hat{u}_h}, \mathsf{U}_{\hat{u}_h}) + (\boldsymbol{Q}_{\mathsf{U}_f, 0}, \boldsymbol{G}_{\mathsf{U}_f, 0}, \mathsf{U}_f),$$

where, on the element $K \in \Omega_h$, for any $\mu \in L^2(\partial K)$ and $f \in L^2(K)$, the functions $\mathsf{U}_\mu, \mathsf{U}_f \in W(K)$ *are the solutions of the local problems*

$$(\mathsf{a}\,\boldsymbol{G}_{\mathsf{U}_\mu,\mu}, \boldsymbol{G}_{w,0})_K + \langle \tau(\mathsf{U}_\mu - \mu), w\rangle_{\partial K} = 0 \qquad \forall w \in W(K),$$
$$(\mathsf{a}\,\boldsymbol{G}_{\mathsf{U}_f,0}, \boldsymbol{G}_{w,0})_K + \langle \tau(\mathsf{U}_f), w\rangle_{\partial K} = (f, w)_K \quad \forall w \in W(K),$$

respectively. The function $\hat{u}_h$ is the element of $M_h(u_D)$ such that

$$a_h(\hat{u}_h, \mu) = \ell_h(\mu) \quad \forall\ \mu\ \in M_h(0),$$

where $a_h(\mu, \lambda) := -\langle \mu, \hat{\boldsymbol{Q}}_{\mathsf{U}_\lambda,\lambda} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}$, and $\ell_h(\mu) := \langle \mu, \hat{\boldsymbol{Q}}_{\mathsf{U}_f,0} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}$. Moreover,

$$a_h(\mu, \lambda) = (\mathsf{a}\boldsymbol{G}_{\mathsf{U}_\mu,\mu}, \boldsymbol{G}_{\mathsf{U}_\lambda,\lambda})_{\partial\Omega_h} + \langle \mathsf{U}_\mu - \mu, \tau(\mathsf{U}_\lambda - \lambda)\rangle_{\partial\Omega_h}, \quad \ell_h(\mu) = (f, \mathsf{U}_\mu),$$

and $a_h(\cdot,\cdot)$ is symmetric and positive definite on $M_h(0) \times M_h(0)$. Thus, $\hat{u}_h$ minimizes the functional $J_h(\mu) := \frac{1}{2}a_h(\mu, \mu) - \ell_h(\mu)$ over $M_h(u_D)$.

Compact Formulation

Finally, we display the compact form of this formulation of the HDG method. We have that $(\boldsymbol{q}_h, \boldsymbol{g}_h) = (\mathbf{Q}_{u_h,\hat{u}_h}, \mathbf{G}_{u_h,\hat{u}_h})$ where $(u_h, \hat{u}_h) \in W_h \times M_h(u_D)$ is the solution of

$$(\mathsf{a}\,\mathbf{G}_{u_h,\hat{u}_h}, \mathbf{G}_{w,\mu})_{\Omega_h} + \langle \tau(u_h - \hat{u}_h), w - \mu\rangle_{\partial\Omega_h} = (f, w)_{\Omega_h}\ \forall (w, \mu) \in W_h \times M_h(0). \tag{6}$$

In other words, $(u_h, \hat{u}_h)$ is the only minimum over $W_h \times M_h(0)$ of the functional

$$J_h(w, \mu) := \frac{1}{2}\{(\mathsf{a}\,\mathbf{G}_{w,\mu}, \mathbf{G}_{u,\mu})_{\Omega_h} + \langle \tau(w - \mu), w - \mu\rangle_{\partial\Omega_h}\} - (f, w)_{\Omega_h}.$$

This is exactly the minimization problem (5).

5 Using Neumann Instead of Dirichlet Boundary Conditions

In the previous two sections, we have shown how a characterization of the exact solution can be used to *generate* HDG methods. Here we show how a different characterization of the exact solution can be used to produce a different *static condensation*, that is, a different way of *implementing*, an already known HDG method.

We proceed as follows. First, we present a characterization of the exact solution which uses Neumann boundary-value problems *instead* of the Dirichlet boundary-value problems to define the local problems. Then, we consider some HDG methods devised in the previous sections and show how a discrete version of the new characterization of the exact solution is nothing but a new way of implementing them. The resulting form of the HDG method has already been used in the work on multiscale methods in [50]. Recently, two different ways of statically condensing the very same method were proposed in [49].

The idea of using different characterizations of the exact solution to devise HDG methods was introduced back in 2009 in [17] where four different ways were presented to devise HDG methods for the vorticity-velocity-pressure formulation of the Stokes system, as the exact solution could be characterized in terms of four different local problems and transmissions conditions. Just as it happens with the exact solution, the very same HDG method could be obtained by using any of the four ways. In other words, the HDG method could be *hybridized* and then *statically condensed* in each of the above-mentioned four different manners.

5.1 A Second Characterization of the Exact Solution

Let us then show how to use local Neumann boundary-value problems to obtain a characterization of the exact solution.

Suppose that, for every element $K \in \Omega_h$, we define $(\mathbf{Q}, \mathsf{U})$ as the solution of the local problem

$$
\begin{aligned}
\mathsf{c}\,\mathbf{Q} + \nabla \mathsf{U} &= 0 && \text{in } K,\\
\nabla\cdot\mathbf{Q} &= f + \{\langle \hat{\boldsymbol{q}}\cdot\boldsymbol{n}, 1\rangle_{\partial K} - (f,1)_K\}/|K| && \text{in } K,\\
\mathbf{Q}\cdot\boldsymbol{n} &= \hat{\boldsymbol{q}}\cdot\boldsymbol{n} && \text{on } \partial K,\\
(\mathsf{U}, 1)_K &= (\overline{u}, 1)_K,
\end{aligned}
$$

where we want the function $\hat{\boldsymbol{q}}$, which has a single-valued normal component, and the constant $\overline{u}$, to be such $(\boldsymbol{q}, u) = (\mathbf{Q}, \mathsf{U})$ on K. This happens *if and only if* $\hat{\boldsymbol{q}}$ and $\overline{u}$ satisfy the equations

$$
\begin{aligned}
[\![\mathsf{U}]\!] &= 0 && \text{for } F \in \mathcal{F}_h^i,\\
\langle \hat{\boldsymbol{q}}\cdot\boldsymbol{n}, 1\rangle_{\partial K} &= (f,1)_K && \text{for } K \in \mathcal{T}_h,\\
\mathsf{U} &= u_D && \text{for } F \in \mathcal{F}_h^\partial.
\end{aligned}
$$

Note that we have to provide the average to U on the element, $\overline{u}$, otherwise the solution U is not uniquely determined. Note also that, we have had to add an additional term to the right-hand side of the second equation in order to ensure that the local problem has a solution for *any* boundary data $\hat{\boldsymbol{q}}\cdot\boldsymbol{n}$. As a consequence, we have to make sure that such term is *zero*. This explains why the global problem consists not only of transmission and boundary conditions, as in the case of Dirichlet boundary-value local problems.

If we now separate the influence of $\hat{\boldsymbol{q}}, \overline{u}$ and f, we readily get the following characterization of the exact solution.

Theorem 9 (Characterization of the Exact Solution) *We have that*

$$(\boldsymbol{q}, u) = (\boldsymbol{Q}, \mathsf{U}) = (\boldsymbol{Q}_{\hat{\boldsymbol{q}}}, \mathsf{U}_{\hat{\boldsymbol{q}}}) + (\boldsymbol{0}, \overline{u}) + (\boldsymbol{Q}_f, \mathsf{U}_f),$$

where $(\boldsymbol{Q}_{\hat{\boldsymbol{q}}}, \mathsf{U}_{\hat{\boldsymbol{q}}})$ *and* $(\boldsymbol{Q}_f, \mathsf{U}_f)$ *are the solution of the local problems*

$$\begin{aligned}
\mathsf{c}\,\boldsymbol{Q}_{\hat{\boldsymbol{q}}} + \nabla \mathsf{U}_{\hat{\boldsymbol{q}}} &= 0 && \text{in } K, & \mathsf{c}\,\boldsymbol{Q}_f + \nabla \mathsf{U}_f &= 0 && \text{in } K,\\
\nabla \cdot \boldsymbol{Q}_{\hat{\boldsymbol{q}}} &= \langle \hat{\boldsymbol{q}} \cdot \boldsymbol{n}, 1\rangle_{\partial K}/|K| && \text{in } K, & \nabla \cdot \boldsymbol{Q}_f &= f - (f,1)_K/|K| && \text{in } K,\\
\boldsymbol{Q}_{\hat{\boldsymbol{q}}} \cdot \boldsymbol{n} &= \hat{\boldsymbol{q}} \cdot \boldsymbol{n} && \text{on } \partial K, & \boldsymbol{Q}_f \cdot \boldsymbol{n} &= 0 && \text{on } \partial K,\\
(\mathsf{U}_{\hat{\boldsymbol{q}}}, 1)_K &= 0, && & (\mathsf{U}_f, 1)_K &= 0.
\end{aligned}$$

where the functions $\hat{\boldsymbol{q}} \cdot \boldsymbol{n}$ *and* $\overline{u}$ *are determined as the solution of the equations*

$$\begin{aligned}
-[\![\mathsf{U}_{\hat{\boldsymbol{q}}}]\!] - [\![\overline{u}]\!] &= [\![\mathsf{U}_f]\!] && \text{on } \mathcal{F}_h^i,\\
\langle \hat{\boldsymbol{q}} \cdot \boldsymbol{n}, 1\rangle_{\partial K} &= (f,1)_K && \text{for } K \in \mathcal{T}_h,\\
\mathsf{U}_{\hat{\boldsymbol{q}}} + \overline{u} &= -\mathsf{U}_f + u_D && \text{on } \mathcal{F}_h^\partial.
\end{aligned}$$

5.2 An Example

In the case of our one-dimensional example, this result reads as follows. We have that

$$(\boldsymbol{q}, u) = (\mathbf{Q}_{\hat{\boldsymbol{q}}}, \mathsf{U}_{\hat{\boldsymbol{q}}}) + (\boldsymbol{0}, \overline{u}) + (\mathbf{Q}_f, \mathsf{U}_f),$$

where

$$\begin{aligned}
\mathsf{c}\,\mathbf{Q}_{\hat{\boldsymbol{q}}} + \frac{d}{dx}\mathsf{U}_{\hat{\boldsymbol{q}}} &= 0 && \text{in } (x_{i-1}, x_i), & \mathsf{c}\,\mathbf{Q}_f + \frac{d}{dx}\mathsf{U}_f &= 0 && \text{in } (x_{i-1}, x_i),\\
\frac{d}{dx}\mathbf{Q}_{\hat{\boldsymbol{q}}} &= \frac{1}{h_i}(\hat{\boldsymbol{q}}_i - \hat{\boldsymbol{q}}_{i-1}) && \text{in } (x_{i-1}, x_i), & \frac{d}{dx}\mathbf{Q}_f &= f - \frac{1}{h_i}\int_{x_{i-1}}^{x_i} f && \text{in } (x_{i-1}, x_i),\\
\mathbf{Q}_{\hat{\boldsymbol{q}}} \cdot \boldsymbol{n} &= \hat{\boldsymbol{q}} \cdot \boldsymbol{n} && \text{on } \{x_{i-1}, x_i\}, & \mathbf{Q}_f \cdot \boldsymbol{n} &= 0, && \text{on } \{x_{i-1}, x_i\},\\
\int_{x_{i-1}}^{x_i} \mathsf{U}_{\hat{\boldsymbol{q}}} &= 0, && & \int_{x_{i-1}}^{x_i} \mathsf{U}_f &= 0,
\end{aligned}$$

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and where the functions $\hat{\boldsymbol{q}}$ and $\overline{u}$ are the solution of

$$\begin{aligned}
\mathsf{U}_{\hat{\boldsymbol{q}}}(x_i^+) - \mathsf{U}_{\hat{\boldsymbol{q}}}(x_i^-) + \overline{u}_{i+1/2} - \overline{u}_{i-1/2} &= -\mathsf{U}_f(x_i^+) + \mathsf{U}_f(x_i^-) && \text{for } i = 1, \ldots, N-1,\\
\hat{\boldsymbol{q}}_i - \hat{\boldsymbol{q}}_{i-1} &= \int_{x_{i-1}}^{x_i} f && \text{for } i = 1, \ldots, N-1,\\
\mathsf{U}_{\hat{\boldsymbol{q}}}(x_0^+) + \overline{u}_{1/2} &= -\mathsf{U}_f(x_0^+) + u_D(x_0),\\
\mathsf{U}_{\hat{\boldsymbol{q}}}(x_N^-) + \overline{u}_{N-1/2} &= -\mathsf{U}_f(x_0^+) + u_D(x_N).
\end{aligned}$$

Since the solution of the local problems are

$$\begin{aligned}
\mathbf{Q}_{\hat{\boldsymbol{q}}}(x) &= \varphi_i(x)\hat{\boldsymbol{q}}_i + \varphi_{i-1}(x)\hat{\boldsymbol{q}}_{i-1}, & \mathbf{Q}_f(x) &= -\mathsf{c}^{-1}\int_{x_{i-1}}^{x_i} \mathsf{G}_x^i(x,s) f(s)\, ds,\\
\mathsf{U}_{\hat{\boldsymbol{q}}}(x) &= \frac{\mathsf{c}\, h_i}{6}\{\psi_i(x)\hat{\boldsymbol{q}}_i - \psi_{i-1}(x)\hat{\boldsymbol{q}}_{i-1}\} & \mathsf{U}_f(x) &= \int_{x_{i-1}}^{x_i} \mathsf{G}^i(x,s) f(s)\, ds.
\end{aligned}$$

where G^i is the Green's function of the second local problem, namely,

$$G^i(x,s) := \begin{cases} \frac{\mathsf{c}\, h_i}{6}\,[1 - 3\varphi_i^2(s) - 3\varphi_{i-1}^2(x)] & \text{if } x_{i-1} \leqslant s \leqslant x,\\ \frac{\mathsf{c}\, h_i}{6}\,[1 - 3\varphi_i^2(x) - 3\varphi_{i-1}^2(s)] & \text{if } x \leqslant s \leqslant x_i. \end{cases}$$

and $\psi_i := 1 - 3\varphi_i^2$, and where the functions $\hat{\boldsymbol{q}}$ and $\overline{u}$ are the solution of

$$\begin{aligned}
\frac{\mathsf{c}\, h_i}{6}(\hat{\boldsymbol{q}}_{i-1} + 2\,\hat{\boldsymbol{q}}_i) + \frac{\mathsf{c}\, h_{i+1}}{6}(2\,\hat{\boldsymbol{q}}_i + \hat{\boldsymbol{q}}_{i+1})&\\
+ \overline{u}_{i+1/2} - \overline{u}_{i-1/2} &= -\mathsf{U}_f(x_i^+) + \mathsf{U}_f(x_i^-) && \text{for } i = 1, \ldots, N-1,\\
\hat{\boldsymbol{q}}_i - \hat{\boldsymbol{q}}_{i-1} &= \int_{x_{i-1}}^{x_i} f && \text{for } i = 1, \ldots, N-1,\\
\frac{\mathsf{c}\, h_1}{6}(2\hat{\boldsymbol{q}}_0 + \hat{\boldsymbol{q}}_1) + \overline{u}_{1/2} &= -\mathsf{U}_f(x_0^+) + u_D(x_0),\\
\frac{\mathsf{c}\, h_N}{6}(\hat{\boldsymbol{q}}_{N-1} - 2\hat{\boldsymbol{q}}_N) - \overline{u}_{N-1/2} &= \mathsf{U}_f(x_N^-) - u_D(x_N).
\end{aligned}$$

5.3 *Another Static Condensation of Known HDG Methods*

Let us consider the HDG methods introduced in Sect. 3. Next, we show that those methods can be statically condensed in the way suggested by our new characterization of the exact solution.

5.3.1 Rewriting the Compact Formulation Based on the Numerical Traces

First, we rewrite them in such a way that the numerical trace $\hat{\boldsymbol{q}}_h$, and not $\hat{u}_h$, is an independent unknown. We can do that very easily if we use the compact formulation of those methods based on the numerical traces, (3). It states that the approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, u_h, \hat{u}_h) \in \boldsymbol{V}_h \times W_h \times M_h(u_D)$ satisfying the equations

$$
\begin{aligned}
(\mathsf{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h} && \forall w \in W_h,\\
\hat{\boldsymbol{q}}_h\cdot\boldsymbol{n} := \boldsymbol{q}_h\cdot\boldsymbol{n} + \tau(u_h - \hat{u}_h) & && \text{on } \partial\Omega_h,\\
\langle \mu, \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0 && \forall \mu \in M_h(0).
\end{aligned}
$$

Now, if we take the stabilization function $\tau(\cdot)$ to be the simple multiplication by the scalar function τ, we have that

$$
\hat{u}_h = u_h + \tau^{-1}(\boldsymbol{q}_h\cdot\boldsymbol{n} - \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}) \qquad \text{on } \partial\Omega_h.
$$

If the local space $\boldsymbol{V}(K) \times W(K)$ is such that, for each face F of the element K,

$$
\begin{aligned}
\boldsymbol{V}(K)\cdot\boldsymbol{n}|_F &\subset M(F),\\
W(K)|_F &\subset M(F),
\end{aligned}
$$

and take τ to be constant on each face of the triangulation, we have that $\hat{\boldsymbol{q}}_h$ belongs to the space

$$
\boldsymbol{N}_h := \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{F}_h) : \ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} \in M(\partial K), \ [\![\boldsymbol{v}]\!] = 0 \text{ on } \mathcal{F}_h^i\}.
$$

We can thus rewrite the HDG method as follows. The approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, u_h, \hat{\boldsymbol{q}}_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{N}_h$ satisfying the equations

$$
\begin{aligned}
(\mathsf{c}\,\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
-(\boldsymbol{q}_h, \nabla w)_{\Omega_h} + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle_{\partial\Omega_h} &= (f, w)_{\Omega_h} && \forall w \in W_h,\\
\hat{u}_h = u_h + \tau^{-1}(\boldsymbol{q}_h\cdot\boldsymbol{n} - \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}) & && \text{on } \partial\Omega_h,\\
\langle \hat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle u_D, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega} && \forall \boldsymbol{v} \in \boldsymbol{N}_h.
\end{aligned}
$$

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Note that the last equation enforces both the single-valuedness of $\hat{u}_h$ as well as the Dirichlet boundary conditions of the model problem (1).

5.3.2 The New Static Condensation

So, suppose that, for every element $K \in \Omega_h$, we define $(\mathbf{Q}, \mathsf{U}) \in \boldsymbol{V}(K) \times W(K)$ to be the solution of the local problem

$$\begin{aligned}
(\mathsf{c}\,\mathbf{Q}, \boldsymbol{v})_K - (\mathsf{U}, \nabla\cdot\boldsymbol{v})_K + \langle \hat{\mathsf{U}}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
-(\mathbf{Q}, \nabla w)_K + \langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w - \overline{w}\rangle_{\partial K} &= (f, w-\overline{w})_{\partial K} && \forall w \in W(K),\\
\hat{\mathsf{U}} &:= \mathsf{U} + \tau^{-1}(\mathbf{Q} - \hat{\boldsymbol{q}}_h)\cdot\boldsymbol{n} && \text{on } \partial K,\\
(\mathsf{U}, 1)_K &= (\overline{u}_h, 1)_K,
\end{aligned}$$

where $\overline{w}|_K := (w,1)_K/|K|$, and where we want to take $\hat{\boldsymbol{q}}_h \in \boldsymbol{N}_h$ and the piecewise constant function $\overline{u}_h$ such that $(\boldsymbol{q}_h, u_h) = (\mathbf{Q}, \mathsf{U})$. Clearly, this happens if we have that $(\hat{\boldsymbol{q}}_h, \overline{u}_h)$ is the solution of the global problem

$$\begin{aligned}
\langle \boldsymbol{v}\cdot\boldsymbol{n}, \hat{\mathsf{U}}\rangle_{\partial\Omega_h} &= \langle \boldsymbol{v}\cdot\boldsymbol{n}, u_D\rangle_{\partial\Omega} && \forall \boldsymbol{v} \in \boldsymbol{N}_h,\\
\langle \hat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, 1\rangle_{\partial K} &= (f, 1)_{\partial K} && \forall\, K \in \Omega_h.
\end{aligned}$$

Separating the influence of $\hat{\boldsymbol{q}}_h$ from that of $\overline{u}_h$ and f, we obtain the following, new static condensation of the HDG method. In what follows, $\overline{W}_h$ denotes the space of real-valued functions which are constant on each element $K \in \Omega_h$.

Theorem 10 (New Static Condensation of HDG Methods) *The approximate solution of the HDG method is*

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{Q}, \mathsf{U}) = (\boldsymbol{Q}_{\hat{\boldsymbol{q}}_h}, \mathsf{U}_{\hat{\boldsymbol{q}}_h}) + (\boldsymbol{0}, \overline{u}_h) + (\boldsymbol{Q}_f, \mathsf{U}_f),$$

where, for each element $K \in \Omega_h$, for any $\boldsymbol{\eta} \in \boldsymbol{L}^2(\partial K)$, the function $(\boldsymbol{Q}_{\boldsymbol{\eta}}, \mathsf{U}_{\boldsymbol{\eta}}) \in \boldsymbol{V}(K) \times W(K)$ is the solution of the local problem

$$\begin{aligned}
(\mathsf{c}\,\boldsymbol{Q}_{\boldsymbol{\eta}}, \boldsymbol{v})_K - (\mathsf{U}_{\boldsymbol{\eta}}, \nabla\cdot\boldsymbol{v})_K + \langle \hat{\mathsf{U}}_{\boldsymbol{\eta}}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
-(\boldsymbol{Q}_{\boldsymbol{\eta}}, \nabla w)_K + \langle \boldsymbol{\eta}\cdot\boldsymbol{n}, w - \overline{w}\rangle_{\partial K} &= 0 && \forall w \in W(K),\\
\hat{\mathsf{U}}_{\boldsymbol{\eta}} &:= \mathsf{U}_{\boldsymbol{\eta}} + \tau^{-1}(\boldsymbol{Q}_{\boldsymbol{\eta}} - \boldsymbol{\eta})\cdot\boldsymbol{n} && \textit{on } \partial K,\\
(\mathsf{U}_{\boldsymbol{\eta}}, 1)_K &= 0,
\end{aligned}$$

and, for any $f \in L^2(K)$*, the function* $(\boldsymbol{Q}_f, \mathsf{U}_f) \in \boldsymbol{V}(K) \times W(K)$ *is the solution of the local problem*

$$\begin{aligned}
(\mathsf{c}\,\boldsymbol{Q}_f, \boldsymbol{v})_K - (\mathsf{U}_f, \nabla\cdot\boldsymbol{v})_K + \langle \hat{\mathsf{U}}_f, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} &= 0 && \forall \boldsymbol{v} \in \boldsymbol{V}(K),\\
-(\boldsymbol{Q}_f, \nabla w)_K &= (f, w - \overline{w})_{\partial K} && \forall w \in W(K),\\
\hat{\mathsf{U}}_f &:= \mathsf{U}_f + \tau^{-1}\boldsymbol{Q}_f\cdot\boldsymbol{n} && \text{on } \partial K,\\
(\mathsf{U}_f, 1)_K &= 0,
\end{aligned}$$

and where $(\hat{\boldsymbol{q}}_h, \overline{u}_h) \in \boldsymbol{N}_h \times \overline{W}_h$ *is the solution of the global problem*

$$\begin{aligned}
a_h(\hat{\boldsymbol{q}}_h, \boldsymbol{\nu}) + b_h(\overline{u}_h, \boldsymbol{\nu}) &= \ell_h(\boldsymbol{\nu}) - \langle u_D, \boldsymbol{\nu}\cdot\boldsymbol{n}\rangle_{\partial\Omega} && \forall \boldsymbol{\nu} \in \boldsymbol{N}_h,\\
b_h(\overline{\omega}, \hat{\boldsymbol{q}}_h) &= (f, \overline{\omega})_{\Omega_h} && \forall \overline{\omega} \in \overline{W}_h,
\end{aligned}$$

where

$$a_h(\boldsymbol{\eta}, \boldsymbol{\nu}) := -\langle \boldsymbol{\nu}\cdot\boldsymbol{n}, \hat{\mathsf{U}}_{\boldsymbol{\eta}}\rangle_{\partial\Omega_h}, \quad b_h(\overline{\omega}, \boldsymbol{\nu}) := -\langle \boldsymbol{\nu}\cdot\boldsymbol{n}, \overline{\omega}\rangle_{\partial\Omega_h}, \quad \ell_h(\boldsymbol{\nu}) := \langle \boldsymbol{\nu}\cdot\boldsymbol{n}, \hat{\mathsf{U}}_f\rangle_{\partial\Omega_h}.$$

Moreover,

$$a_h(\boldsymbol{\eta}, \boldsymbol{\nu}) = (\mathsf{c}\,\boldsymbol{Q}_{\boldsymbol{\eta}}, \boldsymbol{Q}_{\boldsymbol{\nu}})_{\partial\Omega_h} + \langle (\boldsymbol{Q}_{\boldsymbol{\eta}} - \boldsymbol{\eta})\cdot\boldsymbol{n}, \tau^{-1}(\boldsymbol{Q}_{\boldsymbol{\nu}} - \boldsymbol{\nu})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}, \quad \ell_h(\boldsymbol{\nu}) = (f, \mathsf{U}_{\boldsymbol{\nu}})_{\Omega_h},$$

and $\hat{\boldsymbol{q}}_h$ *minimizes the complementary energy functional*

$$\mathcal{J}_h(\boldsymbol{\nu}) := \frac{1}{2}a_h(\boldsymbol{\nu}, \boldsymbol{\nu}) - \ell_h(\boldsymbol{\nu}) + \langle u_D, \boldsymbol{\nu}\cdot\boldsymbol{n}\rangle_{\partial\Omega},$$

over the functions $\boldsymbol{\nu} \in \boldsymbol{N}_h$ *such that* $b_h(\overline{\omega}, \boldsymbol{\nu}) = (f, \overline{\omega})_{\Omega_h}\ \forall \overline{\omega} \in \overline{W}_h$*.*

The proof of this result goes along the very same lines of the proof of the characterization Theorem 5.

5.3.3 The Stabilized Mixed Compact Formulation

Let us end this section by displaying the compact formulation of the method obtained when we eliminate the numerical trace $\hat{u}_h$. Proceeding as for the first characterization, we can obtain that the approximate solution given by the HDG method is the function $(\boldsymbol{q}_h, u_h, \hat{\boldsymbol{q}}_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{N}_h$ satisfying the equations

$$A_h(\boldsymbol{q}_h, \boldsymbol{v}) + S_h(\boldsymbol{q}_h, \hat{\boldsymbol{q}}_h; \boldsymbol{v}, \boldsymbol{\nu}) + B_h(u_h; \boldsymbol{v}, \boldsymbol{\nu}) = -\langle u_D, \boldsymbol{\nu}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}, \tag{7a}$$

$$-B_h(w; \boldsymbol{q}_h, \hat{\boldsymbol{q}}_h) = (f, w)_{\Omega_h}, \tag{7b}$$

for all $(\boldsymbol{v}, w, \boldsymbol{\nu}) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{N}_h$, where

$$A_h(\boldsymbol{p}, \boldsymbol{v}) := (\mathrm{c}\boldsymbol{p}, \boldsymbol{v})_{\Omega_h} \qquad \forall \boldsymbol{p}, \boldsymbol{v} \in \boldsymbol{V}_h, \tag{7c}$$

$$B_h(w; \boldsymbol{v}, \boldsymbol{\nu}) := (\nabla w, \boldsymbol{v})_{\Omega_h} - \langle w, \boldsymbol{\nu} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} \qquad \forall (\boldsymbol{v}, w, \boldsymbol{\nu}) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{N}_h, \tag{7d}$$

$$S_h(\boldsymbol{p}, \boldsymbol{\eta}; \boldsymbol{v}, \boldsymbol{\nu}) := \langle (\boldsymbol{p} - \boldsymbol{\eta}) \cdot \boldsymbol{n}, \tau^{-1}(\boldsymbol{v} - \boldsymbol{\nu}) \cdot \boldsymbol{n} \rangle_{\partial\Omega_h} \; \forall (\boldsymbol{p}, \boldsymbol{\eta}), (\boldsymbol{v}, \boldsymbol{\nu}) \in \boldsymbol{V}_h \times \boldsymbol{N}_h. \tag{7e}$$

As a consequence, the solution $(\boldsymbol{q}_h, \hat{\boldsymbol{q}}_h) \in \boldsymbol{V}_h \times \boldsymbol{N}_h$ minimizes the complementary energy functional

$$\mathcal{J}_h(\boldsymbol{v}, \boldsymbol{\nu}) := \frac{1}{2}\{A_h(\boldsymbol{v}, \boldsymbol{v}) + S_h(\boldsymbol{v}, \boldsymbol{\nu}; \boldsymbol{v}, \boldsymbol{\nu})\} + \langle u_D, \boldsymbol{\nu} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}$$

over all functions $(\boldsymbol{v}, \mu)$ in $\boldsymbol{V}_h \times M_h(u_D)$ such that $B_h(w; \boldsymbol{v}, \boldsymbol{\nu}) = (f, w)\ \forall w \in W_h$.

6 Building Bridges and Constructing Methods

Here, we briefly discuss the evolution of the HDG methods. We being by showing that (some of the earliest) HDG methods can be seen as a particular case of the DG methods introduced in 1998 [24] and analyzed in 2000 [4]. We then recall the strong relation between the HDG and the mixed methods, already pointed out in 2009 [33], and show how this relation drove (and is still driving) the development of superconvergent HDG methods. The bridge built in 2014 [14] between the HDG and the so-called staggered discontinuous Galerkin (SDG), a DG method introduced in 2009 [13] and apparently unrelated to the HDG methods, can be seen as part of this development. We discuss the stabilization introduced by Lehrenfeld (and Schöberl) in 2010 [62]. We end by showing that the so-called Weak-Galerkin methods proposed in 2014 [89] and in 2015 [65, 66], are nothing but rewritings of the HDG methods.

6.1 Relating HDG to Old DG Methods

Here, we consider HDG methods whose numerical method defining the local problems is the so-called local discontinuous Galerkin (LDG) method introduced in [24]. The resulting HDG methods are then called the LDG-H methods. For all of them, the stabilization function τ on any face $F \in \mathcal{F}_h$ is a simple multiplication by a constant which we also denote by τ, that is,

$$\hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} := \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau \cdot (u_h - \hat{u}_h) \quad \text{on } \partial\Omega_h.$$

Examples of local spaces, taken from [33], are shown in the table below.

Method	$\boldsymbol{V}(K)$	$W(K)$	$M(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_{k-1}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_{k}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_{k}(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$

In all these cases, we have that the local spaces $\boldsymbol{V}(K) \times W(K)$ are such that, for each face F of the element K,

$$\boldsymbol{V}(K) \cdot \boldsymbol{n}|_F \subset M(F),$$
$$W(K)|_F \subset M(F).$$

This implies that $[\![\boldsymbol{q}_h]\!] \in M_h$ and the transmission condition becomes $[\![\hat{\boldsymbol{q}}_h]\!] = 0$ on $\mathcal{F}_h^i$. This can only hold if and only if, on $\mathcal{F}_h^i$,

$$\hat{u}_h = \frac{\tau^+ {u_h}^+ + \tau^- {u_h}^-}{\tau^+ + \tau^-} + \frac{1}{\tau^+ + \tau^-}\,[\![\boldsymbol{q}_h]\!],$$
$$\hat{\boldsymbol{q}}_h = \frac{\tau^- {\boldsymbol{q}_h}^+ + \tau^+ {\boldsymbol{q}_h}^-}{\tau^+ + \tau^-} + \frac{\tau^+ \tau^-}{\tau^+ + \tau^-}\,[\![u_h]\!].$$

This implies that the DG methods introduced in [24] and analyzed in [4] that have the above choice of numerical traces can be hybridized and then statically condensed. This is why we call these methods the *hybridizable* DG methods.

Note, that none of these LDG-H methods is an LDG method if we take $\tau^\pm \in (0, \infty)$ since for the method to be an LDG method, we must have that $1/(\tau^+ + \tau^-) = 0$. This shows that none of the DG methods considered in [3] is an LDG-H method with finite values of the stabilization function. In fact, these methods can converge faster than any of the DG methods considered therein. For example, in the case in which $\mathsf{c} = \mathsf{Id}$, $\boldsymbol{V}(K) \times W(K) = \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ and $M(F) = \mathcal{P}_k(F)$ this LDG-H method was analyzed in [4], where is was proven that, for arbitrary shape-regular, polyhedral elements, $\boldsymbol{q}_h$ converges with order $k + 1/2$ and u_h with order $k + 1$, for any $k \geqslant 0$, provided τ is of order one. The convergence is in the $L^2(\Omega)$-norm. On the other hand, other LDG-H methods do have the same order of convergence than those considered in [3]. Indeed, by using the same approach in [4], one can easily show that in the case in which $\boldsymbol{V}(K) \times W(K) = \boldsymbol{\mathcal{P}}_{k-1}(K) \times \mathcal{P}_k(K)$ and $M(F) = \mathcal{P}_k(F)$, $\boldsymbol{q}_h$ converges with order k and u_h with order $k + 1$, for any $k \geqslant 1$, provided τ is of order $1/h$. This result holds for meshes made of general shape-regular, polyhedral meshes.

6.2 Relating HDG to Mixed Methods

As pointed out in [33], if the stabilization function τ is taken to be *identically zero* so that $\hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n}$ on $\mathcal{F}_h$, *and* the transmission condition implies that $[\![\hat{\boldsymbol{q}}_h]\!] = 0$ on $\mathcal{F}_h^i$, we recover the so-called (hybridized version of the) mixed methods if the mixed method is used to define the local problems; see also [2]. In the table below, we display the main examples of mixed methods with this property when K is a simplex and we compare it with one of the first HDG methods, the LDG-H method.

Method	$\boldsymbol{V}(K)$	$W(K)$	$M(F)$
RT	$\boldsymbol{\mathcal{P}}_k(K) + \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
LDG-H	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$
BDM	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$

The strong relation between the mixed method and the HDG methods suggested that the HDG methods might share with the mixed methods some of its convergence properties. This was proven to be true for a special LDG-H method obtained by setting $\tau = 0$ on all the faces of the simplex K except one. This method, called the single face-hybridizable (SFH) method, was introduced and analyzed in [30]. Therein, it was shown that the SFH method is strongly related to the RT and BDM mixed methods. Indeed, the bilinear forms $a_h(\cdot,\cdot)$ of the RT, BDM and SFH methods are the same, and the SFH shares with the RT and BDM the same superconvergence properties.

Next, we briefly describe this superconvergence property. For all of the above methods, the local averages of the error $u - u_h$, converge faster than the errors $u - u_h$ and $\boldsymbol{q} - \boldsymbol{q}_h$. As a consequence, we can define, on the each element K, the new approximation $u_h^\star \in W^*(K) := \mathcal{P}_{k+1}(K)$ as the solution of

$$
\begin{aligned}
(\nabla u_h^\star, \nabla w)_K &= -(\mathrm{c}\,\boldsymbol{q}_h, \nabla w)_K \qquad \text{for all } w \in W^*(K),\\
(u_h^\star, 1)_K &= (u_h, 1)_K,
\end{aligned}
$$

Then $u - u_h^\star$ will converge faster than $u - u_h$. The orders of convergence are displayed in the table below; see [30] for the results on the SFH method and [36] for those on the general LGD-H method. The symbol $\star$ indicates that the non-zero values of the stabilization function τ only need to be uniformly bounded by below.

Method	τ	$\boldsymbol{q}_h$	u_h	$\overline{u}_h$	k
RT	0	$k+1$	$k+1$	$k+2$	$\geqslant 0$
SFH	$\star$	$k+1$	$k+1$	$k+2$	$\geqslant 1$
LDG-H	$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$	$\geqslant 1$
BDM	0	$k+1$	k	$k+2$	$\geqslant 2$
LDG-H	$\mathcal{O}(1/h)$	k	$k+1$	$k+1$	$\geqslant 1$

6.3 The SDG Method as a Limit of SFH Methods

In [14], it was proved that the staggered discontinuous Galerkin (SDG) method, originally introduced in the framework of wave propagation in [13], can be obtained as the limit when the non-zero values of the stabilization function of a *special* SFH method goes to infinity. The special SFH method is obtained as follows. The mesh consists of triangles or tetrahedra subdivided into three triangles or four tetrahedra. On the faces of the bigger simplexes, the stabilization function is not zero; it is equal to zero on all the remaining faces.

By building this bridge between the SDG and the SFH methods, the SDG can now be implemented by hybridization and can share the superconvergence properties of the SFH method. Similarly, the SFH method now share the (related but different) superconvergence property of the SDG method.

6.4 Constructing Superconvergent HDG Methods

The first superconvergent HDG method was the SFH method. A systematic approach to uncover superconvergent HDG methods was undertaken in [39] where the following sufficient conditions were found. The space $\boldsymbol{V}(K) \times W(K)$ must have a subspace $\tilde{\boldsymbol{V}}(K) \times \tilde{W}(K)$ satisfying inclusions

$$\begin{aligned} \boldsymbol{\mathcal{P}}_0(K) &\subset \nabla W(K) \subset \tilde{\boldsymbol{V}}(K),\\ \mathcal{P}_0(K) &\subset \nabla \cdot \boldsymbol{V}(K) \subset \tilde{W}(K),\\ \boldsymbol{V}(K)\cdot \boldsymbol{n} &+ W(K) \subset M(\partial K). \end{aligned}$$

and whose orthogonal complement satisfies the identity

$$\tilde{\boldsymbol{V}}^{\perp} \cdot \boldsymbol{n} \oplus \tilde{W}^{\perp} = M(\partial K).$$

Let us present examples taken from [39] in the case in which K is a cube; the first corresponds to the choice $M(F) = Q^k(F)$ and the second to the choice $M(F) = \mathcal{P}_k(F)$.

In the first example, the HDG method denoted by $\mathbf{HDG}^{Q}_{[k]}$ and the mixed method denoted by $\mathbf{TNT}_{[k]}$ are new. The 7-dimensional space $\boldsymbol{H}^k_7(K)$ is obtained by adding a basis function to the space $\boldsymbol{H}^k_6(K)$. The precise description of these spaces can be found in [39] or, better, in [18], where commuting diagrams for the **TNT** elements on cubes were obtained for the DeRham complex.

In the second example, the HDG method denoted by $\mathbf{HDG}^{P}_{[k]}$ is new. In the corresponding table, we abuse the notation slightly to keep it simple. Thus, by $\mathcal{P}^{k+1}(K)\backslash\tilde{\mathcal{P}}^{k+1}(y,z)$ we mean the span of $\{x^\alpha y^\beta z^\gamma : \alpha+\beta+\gamma \leqslant k+1, \alpha > 0\}$.

$M(F) = Q^k(F), k \geqslant 1$		
Method	$\boldsymbol{V}(K)$	$W(K)$
$\mathbf{RT}_{[k]}$	$\mathcal{P}^{k+1,k,k}(K)$	$Q^k(K)$
	$\times\mathcal{P}^{k,k+1,k}(K)$	
	$\times\mathcal{P}^{k,k,k+1}(K)$	
$\mathbf{TNT}_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}^k_7(K)$	$Q^k(K)$
$\mathbf{HDG}^Q_{[k]}$	$\boldsymbol{Q}^k(K) \oplus \boldsymbol{H}^k_6(K)$	$Q^k(K)$

$M(F) = \mathcal{P}_k(F), k \geqslant 1$		
Method	$\boldsymbol{V}(K)$	$W(K)$
$\mathbf{BDFM}_{[k+1]}$	$\mathcal{P}^{k+1}(K)\backslash\tilde{\mathcal{P}}^{k+1}(y,z)$	$\mathcal{P}_k(K)$
	$\times\mathcal{P}^{k+1}(K)\backslash\tilde{\mathcal{P}}^{k+1}(x,z)$	
	$\times\mathcal{P}^{k+1}(K)\backslash\tilde{\mathcal{P}}^{k+1}(x,y)$	
$\mathbf{HDG}^P_{[k]}$	$\boldsymbol{\mathcal{P}}^k(K)$	$\mathcal{P}^k(K)$
c	$\oplus\nabla\times(yz\,\tilde{\mathcal{P}}^k(K),0,0)$	
	$\oplus\nabla\times(0,zx\,\tilde{\mathcal{P}}^k(K),0)$	
$\mathbf{BDM}_{[k]}$	$\boldsymbol{\mathcal{P}}^k(K)$	$\mathcal{P}^{k-1}(K)$
$k \geqslant 2$	$\oplus\nabla\times(0,0,xy\,\tilde{\mathcal{P}}^k(y,z))$	
	$\oplus\nabla\times(0,xz\,\tilde{\mathcal{P}}^k(x,y),0)$	
	$\oplus\nabla\times(yz\,\tilde{\mathcal{P}}^k(x,z),0,0)$	

In [39], many new superconvergence HDG methods were found for simplexes, squares, cubes and prisms. For curved elements, see [40].

6.5 The Lehrenfeld-Schöberl Stabilization Function

Let us recall that the case in which $M(F) := \mathcal{P}_k(K)$ and $\boldsymbol{V}(K)\times W(K) := \boldsymbol{\mathcal{P}}_{k-1}(K)\times \mathcal{P}_k(K)$, and the stabilization function τ is the multiplicative stabilization function, namely,

$$\tau(u_h - \hat{u}_h) := \tau\cdot(u_h - \hat{u}_h),$$

the resulting method is an LDG-H method. Moreover, for arbitrary shape-regular, polyhedral elements, we have that $\boldsymbol{q}_h$ converges with order k and u_h with order $k+1$, for any $k \geqslant 0$, provided τ is of order $1/h$. Since the size of the stiffness matrix of the local problem is proportional to the number of faces of the triangulation times the dimension of the space $M(F)$, a reduction of the space $M(F)$ would result in a smaller global problem. The question is if this is possible to achieve without loosing the convergence properties of the method.

In 2010, Ch. Lehrenfeld (and J. Schöberl) [62, Remark 1.2.4] noted that the answer is affirmative, see also [63], if we modify the above stabilization function