Chapter 2
An Introduction to the Theory of M-Decompositions

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Abstract We provide a short introduction to the theory of M-decompositions in the framework of steady-state diffusion problems. This theory allows us to systematically devise hybridizable discontinuous Galerkin and mixed methods which can be proven to be superconvergent on unstructured meshes made of elements of a variety of shapes. The main feature of this approach is that it reduces such an effort to the definition, for each element K of the mesh, of the spaces for the flux, $V(K)$, and the scalar variable, $W(K)$, which, roughly speaking, can be *decomposed* into suitably chosen orthogonal subspaces related to the space traces on ∂K of the scalar unknown, $M(\partial K)$. We begin by showing how a simple a priori error analysis motivates the notion of an M-decomposition. We then study the main properties of the M-decompositions and show how to actually construct them. Finally, we provide many examples in the two-dimensional setting. We end by briefly commenting on several extensions including to other equations like the wave equation, the equations of linear elasticity, and the equations of incompressible fluid flow.

2.1 Introduction

The theory of M-decompositions has been recently introduced as an effective tool to systematically find the local spaces defining hybridizable discontinuous Galerkin and mixed methods which can be proven to be superconvergent on unstructured

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meshes made of elements of a variety of shapes. By "superconvergent" we mean that they can provide a new approximation, computed in an elementwise manner, which converges optimally and faster than the original approximation.

The general theory of M-decompositions was introduced in [14, 15, 27] in the framework of steady-state diffusion problems, as a refinement of the work done in [22]. Using some of these M-decompositions, new commutative diagrams for the deRham complex were presented in [16]. The extension to the Stokes system of incompressible fluid flow was done in [25], to the Navier-Stokes equations in [24], and to linear elasticity with symmetric approximate stresses in [13]. In this paper, we provide an introduction to the theory of M-decompositions.

We do this for HDG and mixed methods for the following steady-state diffusion problem:

$$\begin{aligned} \mathrm{c}\boldsymbol{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla \cdot \boldsymbol{q} &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded polyhedral domain, c is a uniformly bounded, uniformly positive definite symmetric matrix-valued function, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. The HDG methods have been thoroughly reviewed in [8]. Therein, the M-decompositions were briefly mentioned as a step in the development of the HDG methods. So, this paper can be considered to be a continuation of such review.

Our intention is to introduce the main ideas about M-decompositions as simply as possible; for a brief historical overview of the effort of devising superconvergent methods defined on unstructured meshes, see [27]. The material of this paper is based on three papers on the early development of M-decompositions. The first is the work done in [22], which provides general sufficient conditions for HDG and mixed methods to be superconvergent. The second is the work done in [27], which refines the previous work and introduces a general theory of M-decompositions for steady-state diffusion problems. The third is [14], which is devoted to the actual construction of M-decompositions in two-space dimensions.

The paper is organized as follows. In Sect. 2.2, we begin by placing the appearance of the idea of M-decompositions into historical perspective. In Sect. 2.3, we then introduce the notion of spaces admitting an M-decomposition and show how to use it to define hybridizable discontinuous Galerkin and mixed methods which can be proven to be superconvergent on unstructured meshes made of elements of a variety of shapes. In Sect. 2.4, we display our general construction of spaces admitting an M-decomposition, and in Sect. 2.5, we give concrete examples. We end in Sect. 2.6 by briefly describing past and ongoing extensions of this approach.

2.2 What Motivated the Appearance of the M-Decompositions?

Here, we briefly place the appearance of the M-decompositions into historical perspective. When the first wave of DG methods appeared around the end of last century, they were criticized because they could not be as efficiently implemented and could not provide as accurate approximations as the well-known hybridized version of the mixed methods. The HDG methods were then introduced in order to address the issue of efficient implementation. In addition, as these HDG methods were shown to be closely related to the mixed methods, a systematic effort started to devise HDG methods with the same superconvergence properties of the mixed methods. The theory of M-decompositions appeared as a tool to systematically do this.

2.2.1 DG Methods

To begin our discussion, let us define the DG methods for the model steady-state diffusion problem. Let $\mathcal{T}_h$ be a conforming mesh of Ω made of polygonal (n = 2) or polyhedral (n = 3) elements K. Let $\partial\Omega_h$ denote the set of boundaries ∂K of the elements $K \in \mathcal{T}_h$, $\mathcal{F}_h$ denote the set of faces F of the elements $K \in \mathcal{T}_h$, and $\mathcal{F}(K)$ denote the set of faces F of the element K. As usual, we write $(\eta\,,\,\zeta)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}(\eta,\zeta)_K$, where $(\eta,\zeta)_D$ denotes the integral of $\eta\zeta$ over the domain $D \subset \mathbb{R}^n$. We also write $\langle\eta\,,\,\zeta\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle\eta\,,\,\zeta\rangle_{\partial K}$, where $\langle\eta\,,\,\zeta\rangle_D$ denotes the integral of $\eta\zeta$ over the 1-codimensional domain D. When vector-valued functions are involved, we use a similar notation.

The DG methods seek an approximation to $(u,\boldsymbol{q})$, $(u_h,\boldsymbol{q}_h)$, in the finite dimensional space $W_h \times \boldsymbol{V}_h$, where

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \ \boldsymbol{v}|_K \in \boldsymbol{V}(K), \quad K \in \mathcal{T}_h\},$$
$$W_h := \{w \in L^2(\mathcal{T}_h) : w|_K \in W(K),\ K \in \mathcal{T}_h\},$$

and determine it as the only solution of the following weak formulation:

$$\begin{aligned}(\mathrm{c}\,\boldsymbol{q}_h\,,\,\boldsymbol{v})_{\mathcal{T}_h} - (u_h\,,\,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle\widehat{u}_h\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\ -(\boldsymbol{q}_h\,,\,\nabla w)_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{q}}_h\cdot\nabla\,,\,w\rangle_{\partial\mathcal{T}_h} &= (f\,,\,w)_{\mathcal{T}_h},\end{aligned}$$

for all $(w,\boldsymbol{v},\mu) \in W_h \times \boldsymbol{V}_h$, where the numerical traces $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h\cdot\nabla$ are suitably defined functions of the unknown $(u_h,\boldsymbol{q}_h)$.

In the 2002 unified analysis of the DG methods [2], it was shown that, for elements of general shapes and $\boldsymbol{V}(K) \times W(K) := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$, the best orders of convergence for all the DG methods treated there in were k for the error in the flux $\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}$, which is suboptimal by 1, and $k+1$ for the error in the scalar variable $\|u - u_h\|_{L^2(\Omega)}$, which is optimal. The same results can also be obtained with $\boldsymbol{V}(K) \times W(K) := \boldsymbol{\mathcal{P}}_{k-1}(K) \times \mathcal{P}_k(K)$.

These orders of converge are obtained, in particular, for the following choice of numerical traces:

$$\widehat{u}_h = \begin{cases} \{\!\{u_h\}\!\} - \boldsymbol{C}_{12} \cdot [\![u_h]\!] + C_{22}\,[\![\boldsymbol{q}_h]\!] & \text{in } \mathcal{F}_h \setminus \partial\Omega, \\ g & \text{in } \mathcal{F}_h \cap \partial\Omega, \end{cases}$$

$$\widehat{\boldsymbol{q}}_h = \begin{cases} \{\!\{\boldsymbol{q}_h\}\!\} + \boldsymbol{C}_{12}\,[\![\boldsymbol{q}_h]\!] + C_{11}\,[\![u_h]\!] & \text{in } \mathcal{F}_h \setminus \partial\Omega, \\ \boldsymbol{q}_h + C_{11}(u_h - g)\boldsymbol{n} & \text{in } \mathcal{F}_h \cap \partial\Omega. \end{cases},$$

and C_{11} positive, of order h^{-1}, $\boldsymbol{C}_{12}$ of order one, and $C_{22} = 0$, that is , for the LDG method [9]. When C_{11} and C_{22} are positive and of order one, and $\boldsymbol{C}_{12}$ is also of order one, it was shown in 2000 in [5] that the order of convergence of the flux increases to $k + 1/2$ and that of the scalar variable remains $k + 1$. In 2009 in [18], when the elements are restricted to be simplexes, it was shown in that, if $C_{11}, 1/C_{11}, C_{22}, 1/C_{22}, |\boldsymbol{C}_{12}|$ are positive and uniformly bounded, the order of the flux and that of the scalar variable are both $k + 1$ and that the error in the local averages superconverges with order $k + 2$, just as happens for the approximations of the well known RT_k and BDM_k mixed methods. This result was obtained by exploiting the relation between these DG methods and the corresponding HDG methods which we introduce next.

2.2.2 HDG Methods

The HDG methods were introduced in 2009 in [19] with the intention of obtaining DG methods for which static condensation was guaranteed. As argued in the 2016 review in [8], this resulted in a significant reduction of the number of globally-coupled degrees of freedom for the DG methods, highlighted the strong link between the HDG methods and the hybridized mixed methods, and led to new DG methods with better accuracy than all previously known DG methods.

The HDG methods seek an approximation to $(u, \boldsymbol{q}, u|_{\mathcal{F}_h})$, $(u_h, \boldsymbol{q}_h, \widehat{u}_h)$, in the finite dimensional space $W_h \times \boldsymbol{V}_h \times M_h$, where

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \ \boldsymbol{v}|_K \in \boldsymbol{V}(K), \ \ K \in \mathcal{T}_h\},$$

$$W_h := \{w \in L^2(\mathcal{T}_h) : w|_K \in W(K), \ K \in \mathcal{T}_h\},$$

$$M_h := \{\mu \in L^2(\mathcal{F}_h) : \mu|_F \in M(F), \ \ F \in \mathcal{F}_h\},$$

and determine it as the only solution of the following weak formulation:

$$(\mathrm{c}\,\boldsymbol{q}_h\,,\,\boldsymbol{v})_{\mathcal{T}_h} - (u_h\,,\,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle\widehat{u}_h\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = 0, \tag{2.1a}$$

$$-(\boldsymbol{q}_h\,,\,\nabla w)_{\mathcal{T}_h} + \langle\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\,,\,w\rangle_{\partial\mathcal{T}_h} = (f\,,\,w)_{\mathcal{T}_h}, \tag{2.1b}$$

$$\langle\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n},\mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} = 0, \tag{2.1c}$$

$$\langle\widehat{u}_h,\mu\rangle_{\partial\Omega} = \langle g,\mu\rangle_{\partial\Omega}, \tag{2.1d}$$

for all $(w,\boldsymbol{v},\mu)\in W_h\times\boldsymbol{V}_h\times M_h$, where

$$\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \alpha(u_h-\widehat{u}_h) \quad \text{on} \quad \partial\mathcal{T}_h. \tag{2.1e}$$

As pointed out in [19], by taking particular choices of the local spaces $\boldsymbol{V}(K)$, $W(K)$ and

$$M(\partial K) := \{\mu\in L^2(\partial K):\ \mu|_F\in M(F) \text{ for all } F\in\mathcal{F}(K)\},$$

and of the *linear local stabilization* function α, different HDG methods are obtained. If we can take α to be zero, we obtain nothing but the well-known hybridized version of the mixed methods. This establishes a strong link between the HDG methods, which use a non-zero stabilization α, and the mixed methods.

It can be shown, see [8, 19], that the very structure of the above weak formulation guarantees that the only globally-coupled degrees of freedom are those of the numerical trace $\widehat{u}_h$. This results in a very efficient implementation of the method which provides a significantly smaller stiffness matrix in comparison to that of all other DG methods.

It can also be shown that the HDG methods are strongly related to previously introduced DG methods. For example, if we take for $\boldsymbol{V}(K)\times W(K) := \boldsymbol{\mathcal{P}}_k(K)\times\mathcal{P}_k(K)$ and $M(F) := \mathcal{P}_k(K)$, and the stabilization function as $\alpha(\mu) := \tau\,\mu$, where τ is a constant on each face, it can be easily shown that the resulting HDG method is nothing but a classic DG methods with the following numerical traces:

$$\widehat{u}_h = \begin{cases} \frac{\tau^+}{\tau^++\tau^-}u_h^+ + \frac{\tau^-}{\tau^++\tau^-}u_h^- + \frac{1}{\tau^++\tau^-}[\![\boldsymbol{q}_h]\!] & \text{in } \mathcal{F}_h\setminus\partial\Omega, \\ g & \text{in } \mathcal{F}_h\cap\partial\Omega, \end{cases}$$

$$\widehat{\boldsymbol{q}}_h = \begin{cases} \frac{\tau^-}{\tau^++\tau^-}\boldsymbol{q}_h^+ + \frac{\tau^+}{\tau^++\tau^-}\boldsymbol{q}_h^- + \frac{\tau^+\tau^-}{\tau^++\tau^-}[\![u_h]\!] & \text{in } \mathcal{F}_h\setminus\partial\Omega, \\ \boldsymbol{q}_h + \tau(u_h-g)\boldsymbol{n} & \text{in } \mathcal{F}_h\cap\partial\Omega. \end{cases},$$

To illustrate the convergence properties of this method, let us consider the model problem

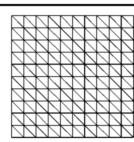
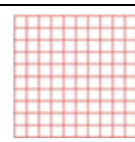
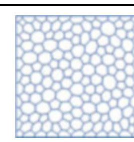
$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = g \quad \text{on } \partial\Omega,$$

where Ω is a unit square, and the exact solution is $u(x, y) = \sin(2\pi x)\sin(2\pi y)$. In the table below, we display a history of convergence for the case $k = 1$ for three different types of meshes and $\tau = 1$. We display the $L^2(\Omega)$-norm of the error between the exact solution u and a local postprocessing u_h^*, see [32, 40, 41], defined on the element K as the polynomial of degree $k+1$ such that

$$(\nabla u_h^*, \nabla w)_K = -(\mathrm{c}\,\boldsymbol{q}_h, \nabla w)_K \quad \forall\, w \in \mathcal{P}_{k+1}(K), \text{ and } (u_h^*, 1)_K = (u_h, 1)_K.$$

For this HDG method, $C_{11} = C_{22} = 1$, $\boldsymbol{C}_{12} = \mathbf{0}$. The results of the first column fully agree with the theoretical predictions in [18] which, for triangular meshes, ensures that the flux converges with order $k+1$ and that the local averages superconverges with order $k+2$; the local postprocessing thus must converge with order $k+2$, as we see in the table. For polygonal meshes, we cannot rely on the theoretical predictions in [5] which only guarantee an order of convergence of the flux of $k+1/2$ and that of the scalar variable is $k+1$.

Thus, we see that the optimal order of convergence for u_h^* of $3 = k+2$ holds only for triangular meshes and deteriorates as the number of sides of the element increases. This raises the question of how to achieve the superconvergence of the local averages independently of the shape of the elements.

						
h	$\Vert u - u_h^\star \Vert_{\mathcal{T}_h}$	Rate	$\Vert u - u_h^\star \Vert_{\mathcal{T}_h}$	Rate	$\Vert u - u_h^\star \Vert_{\mathcal{T}_h}$	Rate
	$\tau = 1$					
0.1	0.15E−2	–	0.83E−2	–	0.52E−2	–
0.05	0.18E−3	3.06	0.16E−2	2.36	0.10E−2	2.34
0.025	0.23E−4	3.03	0.28E−3	2.52	0.19E−3	2.43
0.0125	0.28E−5	**3.02**	0.44E−4	**2.68**	0.35E−4	**2.46**

2.2.3 Local Spaces or Stabilization Functions

The theory of M-decompositions allows us to answer to this question. Roughly speaking, this theory provides an explicit construction of the smallest number of basis functions one has to add to the local spaces of the approximate flux so that the resulting method becomes superconvergent. Once the new local spaces are found, the theory automatically constructs two mixed methods whose local spaces "sandwich" the new found spaces. Thus, we can also consider the theory of M-decompositions as a systematic way of constructing superconvergent mixed methods.

The emphasis of the approach based on M-decompositions is on the construction of the local spaces $\boldsymbol{V}(K) \times W(K)$ and the trace space $M(\partial K)$. It is *not* on the how to determine a stabilization function α which could render the resulting HDG method superconvergent. This second approach represents an complementary alternative to the theory of M-decompositions and is being currently developed. For more details, we refer the reader to before-the-last paragraph of the Introduction in [27].

Here, let us end by briefly mentioning the main contributions to this alternative. Lehrenfeld-Schöberl proposed a new, relatively simple stabilization function back in 2010 in [33, Remark 1.2.4]. The corresponding HDG method was then proven to be superconvergent by Oikawa in 2015 in [34]; see the extension to Stokes in [35]. In a parallel, independent effort, a new, sophisticated stabilization function α was identified in 2015 in [23] which is associated to the hybrid high-order (HHO) methods introduced in 2014 in [29] and in 2015 in [28] (for linear elasticity). See also [36] for an extension to the linear elasticity equations with strong symmetric approximate stresses, and [37] for the Navier-Stokes equations.

2.3 The M-Decompositions

In this section, we show that when the local spaces $\boldsymbol{V}(K) \times W(K)$ admit an $M(\partial K)$-decomposition for every element $K \in \mathcal{T}_h$, the associated HDG or mixed methods are superconvergent on unstructured meshes.

In what follows, to simplify the notation, when there is no possible confusion, we do not indicate the domain on which the functions of a given space are defined. For example, instead of $\boldsymbol{V}(K)$, we simply write $\boldsymbol{V}$.

2.3.1 Definition

To define the M-decomposition of the space

$$\boldsymbol{V} \times W \subset \{\boldsymbol{v} \in \boldsymbol{H}(\text{div}, K) : \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} \in L^2(\partial K)\} \times H^1(K),$$

we need to consider the combined trace operator

$$\begin{aligned} \text{tr} : \boldsymbol{V} \times W \quad &\longrightarrow \quad L^2(\partial K) \\ (\boldsymbol{v}, w) \quad &\longmapsto \quad (\boldsymbol{v} \cdot \boldsymbol{n} + w)|_{\partial K} \end{aligned}$$

where $\boldsymbol{n} : \partial K \to \mathbb{R}^d$ is the unit outward pointing normal field on ∂K.

Definition 2.1 (The M-Decomposition [27]) We say that $\boldsymbol{V} \times W$ admits an M-decomposition when

(a) $\quad \text{tr}(\boldsymbol{V} \times W) \subset M$, and there exists a subspace $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ of $\boldsymbol{V} \times W$ satisfying

(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \widetilde{\boldsymbol{V}} \times \widetilde{W}$,
(c) $\mathrm{tr} : \widetilde{\boldsymbol{V}}^{\perp} \times \widetilde{W}^{\perp} \to M$ is an isomorphism.

Here $\widetilde{\boldsymbol{V}}^{\perp}$ and $\widetilde{W}^{\perp}$ are the $L^2(K)$-orthogonal complements of $\widetilde{\boldsymbol{V}}$ in $\boldsymbol{V}$, and of $\widetilde{W}$ in W, respectively.

Although it can be proven that we must have $\widetilde{W} = \nabla \cdot \boldsymbol{V}$, the space $\widetilde{\boldsymbol{V}}$ is not unique. However, it is always possible to choose $\widetilde{\boldsymbol{V}}$ as indicated in the following result which is expressed in terms of the following space of solenoidal, $\boldsymbol{H}(\mathrm{div}, K)$-bubbles:

$$\boldsymbol{V}_{\mathrm{sbb}} := \{\boldsymbol{v} \in \boldsymbol{V} : \nabla \cdot \boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} = 0\}.$$

Proposition 2.1 (The Canonical M-Decomposition [27]) *If the space $\boldsymbol{V} \times W$ admits an M-decomposition, then it admits an M-decomposition based on the subspaces*

$$\widetilde{\boldsymbol{V}} = \nabla W \oplus \boldsymbol{V}_{\mathrm{sbb}} \qquad \textit{(orthogonal sum)}, \qquad \widetilde{W} = \nabla \cdot \boldsymbol{V}.$$

Of course, it is far from obvious that spaces $\boldsymbol{V} \times W$ admitting M-decompositions can lead to superconvergent HDG and mixed methods. To see that, we need to carry out the error analysis of the methods with the help of a projection we define next.

2.3.2 The HDG-Projection

We define this auxiliary projection in terms of the $L^2(\partial K)-$projection into $M(\partial K)$, which we denote by P_M.

Definition 2.2 (The HDG-Projection [22]) Let $(\boldsymbol{q}, u)$ be smooth enough so that their boundary traces are in $L^2(\partial K)$. Let $\boldsymbol{V} \times W$ admit an M-decomposition. Then, the pair $\Pi_h(\boldsymbol{q}, u) = (\boldsymbol{\Pi}_V \boldsymbol{q}, \Pi_W u) \in \boldsymbol{V} \times W$ defined by the equations

(α) $(\Pi_W u, w)_K = (u, w)_K \qquad \forall w \in \widetilde{W}$,
(β) $(\boldsymbol{\Pi}_V \boldsymbol{q}, \boldsymbol{v})_K = (\boldsymbol{q}, \boldsymbol{v})_K \qquad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}}$,
(γ) $\langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n} + \alpha(\Pi_W u - P_M u), \mu\rangle_{\partial K} = \langle \boldsymbol{q} \cdot \boldsymbol{n}, \mu\rangle_{\partial K} \qquad \forall \mu \in M$,

is the HDG-projection associated to the M-decomposition and to the stabilization operator $\alpha : L^2(\partial K) \to L^2(\partial K)$.

Note that, when the stabilization function α is zero, we obtain nothing but the well-known projection used for the analysis of the mixed methods. The HDG-projection is thus an extension of such projection. Indeed, for any $w \in W$, we have

$$\begin{aligned}(\Pi_W \nabla \cdot \boldsymbol{q}, w)_K &= -(\boldsymbol{q}, \nabla w)_K + \langle \boldsymbol{q} \cdot \boldsymbol{n}, w\rangle_{\partial K}\\ &= -(\boldsymbol{\Pi}_V \boldsymbol{q}, \nabla w)_K + \langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n} + \alpha(\Pi_W u - P_M u), w\rangle_{\partial K}\\ &= (\nabla \cdot \boldsymbol{\Pi}_V \boldsymbol{q}, w)_K + \langle \alpha(\Pi_W u - P_M u), w\rangle_{\partial K},\end{aligned}$$

and if we define $L_W(m)$ as the element of W such that

$$(L_W(m), w)_K = \langle m, w\rangle_{\partial K} \quad \forall\, w \in W,$$

we can write

$$\Pi_W \nabla \cdot \boldsymbol{q} = \nabla \cdot \boldsymbol{\Pi}_V \boldsymbol{q} + L_W(\alpha(\Pi_W u - P_M u)).$$

This extends to our framework the commutativity properties of the projections Π_W and $\boldsymbol{\Pi}_V$ for the mixed methods, that is, for the case in which we can take $\alpha = 0$.

Next, we provide a sufficient condition on the stabilization function α ensuring that the HDG-projection is actually well defined.

Proposition 2.2 (The HDG-Projection [22]) *Let $\boldsymbol{V} \times W$ admit an M-decomposition. Then the auxiliary HDG-projection Π_h is well defined if we take the linear stabilization operator $\alpha : L^2(\partial K) \to L^2(\partial K)$ such that*

$$w \in \widetilde{W}^{\perp}: \quad \langle \alpha(w), w\rangle_{\partial K} = 0 \quad \Longrightarrow \quad w = 0.$$

This result shows that we can take the stabilization function α equal to zero whenever $\widetilde{W}^{\perp} = \{0\}$. In this way, the stabilization function α can be linked to the gap between W and $\widetilde{W} = \nabla \cdot \boldsymbol{V}$. To measure such a gap, we introduce the following number, which is nonnegative because of the inclusion property (b).

Definition 2.3 (The S-Index) The S-index ("S" for stabilization) of the space $\boldsymbol{V} \times W$ is the number

$$I_S(\boldsymbol{V} \times W) := \dim W - \dim \nabla \cdot \boldsymbol{V}.$$

Note that by the inclusion condition (b), $I_S(\boldsymbol{V} \times W)$ is a natural number. It is zero if and only if $\widetilde{W}^{\perp} = \{0\}$ in which case we can take $\alpha = 0$.

Proof (of Proposition 2.2) Let us start by noting that the system defining the projection is square. The number of equations is $\dim \widetilde{\boldsymbol{V}} + \dim \widetilde{W} + \dim M$ and the number of unknowns is $\dim \boldsymbol{V} + \dim W$. Let us show that these numbers coincide. Since $\boldsymbol{V} \times W$ admits an M-decomposition, there are spaces $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$ satisfying property (c), and so

$$\dim M = \dim \widetilde{\boldsymbol{V}}^{\perp} + \dim \widetilde{W}^{\perp}.$$

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This implies that $\dim \widetilde{\boldsymbol{V}} + \dim \widetilde{W} + \dim M = \dim \boldsymbol{V} + \dim W$, and so the system is square.

Now we only have to set $(\boldsymbol{q}, u) = (\boldsymbol{0}, 0)$ and prove that the only solution is the trivial one. In this case, we get that

$$\begin{aligned}(\varPi_W u, w)_K &= 0 \qquad \forall w \in \widetilde{W},\\ (\boldsymbol{\varPi}_V \boldsymbol{q}, \boldsymbol{v})_K &= 0 \qquad \forall \boldsymbol{v} \in \widetilde{\boldsymbol{V}},\\ \langle \boldsymbol{\varPi}_V \boldsymbol{q} \cdot \boldsymbol{n} + \alpha(\varPi_W u), \mu\rangle_{\partial K} &= 0 \qquad \forall \mu \in M,\end{aligned}$$

which means that $\boldsymbol{\varPi}_V \boldsymbol{q} \in \widetilde{\boldsymbol{V}}^\perp$ and that $\varPi_W u \in \widetilde{W}^\perp$. Since, by property (a), $W|_{\partial K} \subset M$, we can take $\mu := \varPi_W u$ in the third equation defining the projection to get

$$\begin{aligned}\langle \alpha(\varPi_W u), \varPi_W u\rangle_{\partial K} &= -\langle \boldsymbol{\varPi}_V \boldsymbol{q} \cdot \boldsymbol{n}, \varPi_W u\rangle_{\partial K}\\ &= (\nabla \cdot \boldsymbol{\varPi}_V \boldsymbol{q}, \varPi_W u)_K + (\boldsymbol{\varPi}_V \boldsymbol{q}, \nabla \varPi_W u)_K\\ &= 0,\end{aligned}$$

by the inclusion properties (b), since $\nabla \cdot \boldsymbol{\varPi}_V \boldsymbol{q} \in \nabla \cdot \boldsymbol{V} \subset \widetilde{W}$ and since $\nabla \varPi_W u \in \nabla W \subset \widetilde{\boldsymbol{V}}$. Therefore, by the assumption on the stabilization function α, it follows that $\varPi_W u = 0$. Finally, by property (a), since $\boldsymbol{V} \cdot \boldsymbol{n}|_{\partial K} \subset M$, we can take $\mu := \boldsymbol{\varPi}_V \boldsymbol{q} \cdot \boldsymbol{n}$ in the third equation defining the projection to get

$$\langle \boldsymbol{\varPi}_V \boldsymbol{q} \cdot \boldsymbol{n}, \boldsymbol{\varPi}_V \boldsymbol{q} \cdot \boldsymbol{n}\rangle_{\partial K} = 0,$$

which implies, by property (c), that $\boldsymbol{\varPi}_V \boldsymbol{q} = \boldsymbol{0}$ since $\boldsymbol{\varPi}_V \boldsymbol{q} \in \widetilde{\boldsymbol{V}}^\perp$. This completes the proof. □

2.3.3 Estimates of the Projection of the Errors

Next, we find the equations of the projection of the errors:

$$\boldsymbol{e_q} := \boldsymbol{\varPi}_V \boldsymbol{q} - \boldsymbol{q}_h, \quad e_u := \varPi_W u - u_h, \quad \boldsymbol{e}_{\widehat{\boldsymbol{q}}} \cdot \boldsymbol{n} := P_M(\boldsymbol{q} \cdot \boldsymbol{n}) - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \quad e_{\widehat{u}} := P_M(u) - \widehat{u}_h.$$

We show that the definition of an M-decomposition and that of the HDG-projection are tailored to the numerical schemes under consideration.

Since the exact solution also satisfies the weak formulation defining the HDG method, we can write that

$$\begin{aligned}(\mathrm{c}\,(\boldsymbol{q} - \boldsymbol{q}_h)\,,\ \boldsymbol{v})_{\mathcal{T}_h} - (u - u_h\,,\ \nabla \cdot \boldsymbol{v})_{\mathcal{T}_h} + \langle u - \widehat{u}_h\,,\ \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial \mathcal{T}_h} &= 0,\\ -(\boldsymbol{q} - \boldsymbol{q}_h\,,\ \nabla w)_{\mathcal{T}_h} + \langle \boldsymbol{q} \cdot \boldsymbol{n} - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\,,\ w\rangle_{\partial \mathcal{T}_h} &= 0,\end{aligned}$$

$$\begin{aligned} \langle \boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &= 0,\\ \langle u - \widehat{u}_h, \mu\rangle_{\partial\Omega} &= 0, \end{aligned}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$, where $\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \alpha(u_h - \widehat{u}_h)$ on $\partial\mathcal{T}_h$. But, we have that

$$\begin{aligned} \langle u - \widehat{u}_h\,,\ \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= \langle e_{\widehat{u}}\,,\ \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} && \text{by property (a)},\\ \langle \boldsymbol{q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\,,\ w\rangle_{\partial\mathcal{T}_h} &= \langle \boldsymbol{e}_{\widehat{\boldsymbol{q}}}\cdot\boldsymbol{n}\,,\ w\rangle_{\partial\mathcal{T}_h} && \text{by property (a)},\\ (u - u_h\,,\ \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} &= (e_u\,,\ \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} && \text{by properties } (\alpha) \text{ and (b)},\\ (\boldsymbol{q} - \boldsymbol{q}_h\,,\ \nabla w)_{\mathcal{T}_h} &= (\boldsymbol{e}_{\boldsymbol{q}}\,,\ \nabla w)_{\mathcal{T}_h} && \text{by properties } (\beta) \text{ and (b)},\\ \boldsymbol{e}_{\widehat{\boldsymbol{q}}}\cdot\boldsymbol{n} &= \boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n} + P_M\alpha(e_u - e_{\widehat{u}}) && \text{on } \partial\mathcal{T}_h, \end{aligned}$$

by property (γ), and so, we get that

$$\begin{aligned} -(e_u\,,\ \nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle e_{\widehat{u}}\,,\ \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= -(\mathrm{c}\,(\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q})\,,\ \boldsymbol{v})_{\mathcal{T}_h},\\ -(\boldsymbol{e}_{\boldsymbol{q}}\,,\ \nabla w)_{\mathcal{T}_h} + \langle \boldsymbol{e}_{\widehat{\boldsymbol{q}}}\cdot\boldsymbol{n}\,,\ w\rangle_{\partial\mathcal{T}_h} &= 0,\\ \langle \boldsymbol{e}_{\widehat{\boldsymbol{q}}}\cdot\boldsymbol{n}, \mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &= 0,\\ \langle e_{\widehat{u}}, \mu\rangle_{\partial\Omega} &= 0, \end{aligned}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$.

We immediately see that if the right-hand side of the first equation is zero, then the all the projection of the errors are zero. This means that all of them are controlled by the size of the approximation error $\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q}$. In particular, the standard energy argument, obtained by setting $(\boldsymbol{v}, w, \mu) := (\boldsymbol{e}_{\boldsymbol{q}}, e_u, e_{\widehat{u}})$ and adding the equations, and noting that $e_{\widehat{u}}|_{\partial\Omega} = 0$, gives that

$$(\mathrm{c}\,\boldsymbol{e}_{\boldsymbol{q}}\,,\ \boldsymbol{e}_{\boldsymbol{q}})_{\mathcal{T}_h} + \langle \alpha(e_u - e_{\widehat{u}})\,,\ e_u - e_{\widehat{u}}\rangle_{\partial\mathcal{T}_h} = -(\mathrm{c}\,(\boldsymbol{q} - \boldsymbol{\Pi}_V\boldsymbol{q})\,,\ \boldsymbol{e}_{\boldsymbol{q}})_{\mathcal{T}_h}.$$

In fact, it is possible to prove the following estimates.

Theorem 2.1 (A Priori Error Estimates) *Suppose that for every $K \in \mathcal{T}_h$, the space $\boldsymbol{V}(K) \times W(K)$ admits an $M(\partial K)$-decomposition and that the stabilization function α satisfies the following properties:*

(i) $w \in \widetilde{W}^{\perp}(K), \quad \langle \alpha(w), w\rangle_{\partial K} = 0 \quad \Longrightarrow \quad w = 0,$
(ii) $\langle \alpha(\mu), \mu\rangle_{\partial K} \geq 0$ *for all* $\mu \in M(\partial K)$,
(iii) $\langle \alpha(\lambda), \mu\rangle_{\partial K} = \langle \lambda, \alpha(\mu)\rangle_{\partial K}$, *for all* $\lambda, \mu \in M(\partial K)$.

Then, we have

$$\|\boldsymbol{e_q}\|_{\mathcal{T}_h} \leq C\,\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\mathcal{T}_h},$$
$$\|e_u\|_{\mathcal{T}_h} \leq C\,H\,\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\mathcal{T}_h},$$

where $H = 1$ *for general polyhedral domains. For convex polyhedral domains, we have that* $H = h$ *provided*

$$\mathcal{P}_0(K) \subset \nabla W(K)\ \forall\ K \in \mathcal{T}_h.$$

2.3.4 Local Postprocessing

Next, we define an elementwise postprocessing u_h^* defined to converge faster than the original approximation u_h; we follow [32, 40, 41]. We take the postprocessing u_h^* in the space

$$W_h^* := \{w \in L^2(\mathcal{T}_h) : w|_K \in W^*(K),\ K \in \mathcal{T}_h\},$$

and define it as follows. On each element $K \in \mathcal{T}_h$, the function u_h^* is the element of $W^*(K)$ such that

$$(\nabla u_h^*, \nabla w)_K = -\,(c\,\boldsymbol{q}_h, \nabla w)_K \quad \forall\ w \in \widetilde{W}^*(K)^{\perp},$$
$$(u_h^*, w)_K = (u_h, w)_K \quad \forall\ w \in \widetilde{W}^*(K).$$

where $W^*(K) = \widetilde{W}^*(K) \oplus \widetilde{W}^*(K)^{\perp}$ and $\widetilde{W}^*(K)$ is any non-trivial subspace of $\widetilde{W}(K)$ containing constant functions. We have the following result which follows directly from the analysis carried out in [22].

Theorem 2.2 *Under the assumptions of the previous result, and if*

$$\mathcal{P}_0(K) \subset \nabla \cdot \boldsymbol{V}(K)\ \forall\ K \in \mathcal{T}_h,$$

then

$$\|u - u_h^*\|_{\mathcal{T}_h} \leq \|\Pi_W u - u_h\|_{\mathcal{T}_h} + C\,h\,(\|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\|_{\mathcal{T}_h} + \inf_{\omega \in W_h^*} \|\nabla(u - \omega)\|_{\mathcal{T}_h}).$$

This result states that, once we find spaces $\boldsymbol{V} \times W$ spaces admitting M-decompositions, we still have to check the conditions

(J.1) $\mathcal{P}_0(K) \subset \nabla \cdot \boldsymbol{V}$,
(J.2) $\mathcal{P}_1(K) \subset W$,

in order to achieve the superconvergence of the elementwise averages and the optimal convergence of the elementwise postprocessing.

It remains to obtain the approximation properties of the HDG-projection. We do that next.

2.3.5 Approximation Properties of the HDG-Projection

Note that, in view of the second equation defining the auxiliary HDG-projection, one might think that its approximation properties depend on the choice of the subspace $\widetilde{\boldsymbol{V}}$. This would be rather unpleasant given that, unlike the subspace $\widetilde{W}$, the subspace $\widetilde{\boldsymbol{V}}$ of an M-decomposition is *not* uniquely defined. Fortunately, this is not so as we see in the next result which is a small variation of a similar result in [22]; for the sake of completeness, we include a proof in the Appendix. To state it, we need to introduce the quantities

$$a_{\widetilde{W}^\perp} := \begin{cases} \inf_{\mu\in\gamma\,\widetilde{W}^\perp\setminus\{0\}} \langle \alpha(\mu), \mu\rangle_{\partial K} / \|\mu\|^2_{\partial K} & \text{if } \widetilde{W}^\perp \neq \{0\}, \\ \infty & \text{if } \widetilde{W}^\perp = \{0\}, \end{cases}$$

and

$$\|\alpha\| := \sup_{\lambda,\mu\in M\setminus\{0\}} \langle \alpha(\lambda), \mu\rangle_{\partial K} / (\|\lambda\|_{\partial K}\|\mu\|_{\partial K}).$$

When $\widetilde{W}^\perp = \{0\}$, that is, when $\widetilde{W} = W$, we take $\alpha := 0$.

In what follows, P_S denotes the $L^2(\Omega)-$projection into the space S. We use this notation for $S := \boldsymbol{V}_h$, $S := W$ and $S := \widetilde{W}$.

Proposition 2.3 (Approximation Properties of the HDG-Projection) *Let $\boldsymbol{V} \times W$ admit an M-decomposition, and let the stabilization function α satisfy the condition*

$$a_{\widetilde{W}^\perp} > 0.$$

Then, we have

$$\begin{aligned} \|\boldsymbol{q} - \boldsymbol{\Pi}_V \boldsymbol{q}\,\|_K \le\; & \|(Id - P_{\boldsymbol{V}})\,\boldsymbol{q}\,\|_K + \mathsf{C}_1\, h_K^{1/2}\, \|((Id - P_{\boldsymbol{V}})\boldsymbol{q})\cdot\boldsymbol{n}\|_{\partial K} \\ & + \mathsf{C}_2\, h_K\, \|(Id - P_{\widetilde{W}})\nabla\cdot\boldsymbol{q}\,\|_K + \mathsf{C}_3\, h_K^{1/2}\, \|(Id - P_W)u\|_{\partial K}, \\ \|u - \Pi_W u\|_K \le\; & \|(Id - P_W)u\|_K + \mathsf{C}_4\, h_K^{1/2}\, \|(Id - P_W)u\|_{\partial K} \\ & + \mathsf{C}_5\, h_K\, \|(Id - P_{\widetilde{W}})\nabla\cdot\boldsymbol{q}\,\|_K, \end{aligned}$$

where $\mathsf{C}_1 := C_{\widetilde{V}^\perp}$ *and*

$$\mathsf{C}_2 := \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}} C_{\widetilde{V}^\perp} \|\alpha\|, \ \mathsf{C}_3 := \big(1 + \frac{\|\alpha\|}{a_{\widetilde{W}^\perp}}\big) C_{\widetilde{V}^\perp} \|\alpha\|, \ \mathsf{C}_4 := \frac{C_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}} \|\alpha\|, \ \mathsf{C}_5 := \frac{C^2_{\widetilde{W}^\perp}}{a_{\widetilde{W}^\perp}}.$$

Here

$$C_{\widetilde{V}^\perp} := \sup_{\boldsymbol{v} \in \widetilde{V}^\perp \setminus \{\boldsymbol{0}\}} h_K^{-1/2} \|\boldsymbol{v}\|_K / \|\boldsymbol{v} \cdot \boldsymbol{n}\|_{\partial K}, \qquad C_{\widetilde{W}^\perp} := \sup_{w \in \widetilde{W}^\perp \setminus \{0\}} h_K^{-1/2} \|w\|_K / \|w\|_{\partial K},$$

Note that the fact that the coercivity constant $a_{\widetilde{W}^\perp}$ is positive implies the property of the stabilization function α used in Proposition 2.2: this is due to the third condition in the definition of M-decomposition. Note also that, if $W = \widetilde{W} = \nabla \cdot \ \boldsymbol{V}$, then $\mathsf{C}_i = 0$ for $i = 2, 3, 4, 5$ since in this case we are taking $\alpha = 0$ and $a_{\widetilde{W}^\perp} = \infty$.

2.4 A Construction of M-Decompositions

Here, we show how to use the notion of M-decompositions to actually construct spaces admitting M-decompositions. To do that, we begin by establishing a characterization of M-decompositions which is going to be the basis for the construction. We then apply it to show, given an element K, a space of traces $M(\partial K)$, and a the space $\boldsymbol{V}_g \times W_g$, how to systematically construct *three* spaces admitting an M-decomposition. One of them generates an HDG method whereas the other two generate mixed methods.

2.4.1 A Characterization of M-Decompositions

We begin by stating the main result of this section, namely, a characterization of the M-decompositions expressed solely in terms of the spaces $\boldsymbol{V} \times W$. Roughly speaking, it states that $\boldsymbol{V} \times W$ admits an M-decomposition if and only if the space M is the orthogonal sum of the traces of the kernels of $\nabla \cdot$ in $\boldsymbol{V}$ and of ∇ in W. It is expressed in terms of a special integer we define next.

Definition 2.4 (The M-Index) The M-index of the space $\boldsymbol{V} \times W$ is the number

$$\begin{aligned} I_M(\boldsymbol{V} \times W) := \ \dim M - \dim\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \ \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \\ - \dim\{w|_{\partial K} : \ w \in W, \nabla w = 0\}. \end{aligned}$$

Theorem 2.3 (A Characterization of M-Decompositions) *For a given space of traces M, the space $\boldsymbol{V} \times W$ admits an M-decomposition if and only if*

(a) $tr(\boldsymbol{V} \times W) \subset M$,
(b) $\nabla W \times \nabla \cdot \boldsymbol{V} \subset \boldsymbol{V} \times W$,
(c) $I_M(\boldsymbol{V} \times W) = 0$.

In this case, we have the so-called the kernels' trace decomposition *identity*

$$M = \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W, \nabla w = 0\},$$

where the sum is orthogonal.

Note that the subspaces $\widetilde{\boldsymbol{V}}$ and $\widetilde{W}$ appearing in the definition of an M-decomposition, which were strongly associated to the very form of the HDG methods under consideration, are not present anymore in this characterization. This suggests that the M-decomposition can be considered to be associated to the operators $(\nabla\cdot, \nabla)$ rather than to a specific numerical method.

Note also that the above result states that, if the space $\boldsymbol{V} \times W$ satisfies the inclusion conditions (a) and (b), we have that

$$M = C_M \oplus \{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W, \nabla w = 0\},$$

for some subspace C_M of M. This means that the dimension of C_M is nothing but $I_M(\boldsymbol{V} \times W)$ and that $\boldsymbol{V} \times W$ admits an M-decomposition if and only if $C_M = \{0\}$, that is, if and only if $I_M(\boldsymbol{V} \times W) = 0$.

2.4.2 The General Construction

Here, we show how to use the above result to construct spaces admitting M-decompositions. We proceed as follows. First, given the element K and the space of traces $M(\partial K)$, we pick our favorite space $\boldsymbol{V}_g \times W_g$ satisfying the inclusion properties (a) and (b) of Theorem 2.3. Then, we construct three of spaces admitting an M-decomposition as follows.

Step 1. We find a space $\delta \boldsymbol{V}_{\text{fillM}}$ such that

(a) $\delta \boldsymbol{V}_{\text{fillM}} \cdot \boldsymbol{n}|_{\partial K} = C_M$,
(b) $\nabla \cdot \delta \boldsymbol{V}_{\text{fillM}} = \{0\}$,
(c) $\dim \delta \boldsymbol{V}_{\text{fillM}} = I_M(\boldsymbol{V}_g \times W_g)$.

Then, we can verify that $(\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}) \times W_g$ admits an M-decomposition.

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Step 2. The space $(\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}) \times \nabla \cdot \boldsymbol{V}_g$ immediately admits an M-decomposition provided

$$\{w|_{\partial K} : w \in W_g, \nabla w = 0\} = \{w|_{\partial K} : w \in \nabla \cdot \boldsymbol{V}_g, \nabla w = 0\}.$$

In this case, we can take the stabilization function α equal to zero and so the corresponding method is a mixed method.

Step 3. Finally, if $W_g = C_W \oplus \nabla \cdot \boldsymbol{V}_g$, we find a space $\delta \boldsymbol{V}_{\text{fillW}}$ such that

(a) $\delta \boldsymbol{V}_{\text{fillW}} \cdot \boldsymbol{n}|_{\partial K} \subset M$,
(b) $\nabla \cdot \delta \boldsymbol{V}_{\text{fillW}} = C_W$,
(c) $\dim \delta \boldsymbol{V}_{\text{fillW}} = I_S(\boldsymbol{V}_g \times W_g)$.

Then we immediately have that $(\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}} \oplus \delta \boldsymbol{V}_{\text{fillW}}) \times W_g$ admits an M-decomposition. Moreover, we can take the stabilization function α equal to zero and so the corresponding method is a mixed method.

We summarize our construction of spaces admitting M-decompositions in Tables 2.1, 2.2 and 2.3.

Table 2.1 Construction of spaces $\boldsymbol{V} \times W$ admitting an M-decomposition, where the space of traces $M(\partial K)$ includes the constants

$\boldsymbol{V}$	W	$\nabla \cdot \boldsymbol{V}$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}} \oplus \delta \boldsymbol{V}_{\text{fillW}}$	W_g (if $\supset \mathcal{P}_0(K)$)	W_g
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}$	W_g (if $\supset \mathcal{P}_0(K)$)	$\subset W_g$
$\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\text{fillM}}$	$\nabla \cdot \boldsymbol{V}_g$ (if $\supset \mathcal{P}_0(K)$)	$\nabla \cdot \boldsymbol{V}_g$

The given space $\boldsymbol{V}_g \times W_g$ satisfies the inclusion properties (a) and (b)

Table 2.2 The properties of the spaces $\delta \boldsymbol{V}$

$\delta \boldsymbol{V}$	$\nabla \cdot \delta \boldsymbol{V}$	$\delta \boldsymbol{V} \cdot \boldsymbol{n}\|_{\partial K}$	$\dim \delta \boldsymbol{V}$
$\delta \boldsymbol{V}_{\text{fillM}}$	$\{\boldsymbol{0}\}$	C_M	$I_M(\boldsymbol{V}_g \times W_g)$
$\delta \boldsymbol{V}_{\text{fillW}}$	C_W	$\subset M$	$I_S(\boldsymbol{V}_g \times W_g)$

The computation of the space C_W is fairly simple and, usually, independent of the shape of the element. In contrast, the computation of the space C_M is the most difficult part of the construction

Table 2.3 The spaces $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ defining the canonical decomposition of each space $\boldsymbol{V} \times W$ in terms of the space $\boldsymbol{V}_g \times W_g$

$\widetilde{\boldsymbol{V}}$	$\widetilde{W}$
$\nabla W_g \oplus \boldsymbol{V}_{g,\text{sbb}}$	W_g
$\nabla W_g \oplus \boldsymbol{V}_{g,\text{sbb}}$	$\nabla \cdot \boldsymbol{V}_g$
$\nabla(\nabla \cdot \boldsymbol{V}_g) \oplus \boldsymbol{V}_{g,\text{sbb}}$	$\nabla \cdot \boldsymbol{V}_g$

Here $\boldsymbol{V}_{g,\text{sbb}} := \{\boldsymbol{v} \in \boldsymbol{V}_g : \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} = 0\}$

2.5 Examples

Here, we give examples of this construction. We only present the spaces that can be concisely described and so we restrict ourselves to the two-dimensional case. First, we show the computation by hand of the whole construction in a very simple case. We then consider triangular, rectangular and quadrilateral elements and show the old and new spaces that result from our construction. Finally, we describe and briefly discuss the case of a general polygonal element.

2.5.1 An Illustration of the Construction

Let us illustrate the general construction just sketched in a very simple case, namely, when K is the unit square and

$$\begin{aligned} M(\partial K) &:= \{\mu \in L^2(\partial K) : \mu|_F \in \mathcal{P}_0(F) \text{ for all faces } F \text{ of } K\}, \\ \boldsymbol{V}_g \times W_g &:= \boldsymbol{\mathcal{P}}_0(K) \times \mathcal{P}_0(K). \end{aligned}$$

Here, $\mathcal{P}_0$ denotes the space constant functions, and $\boldsymbol{\mathcal{P}}_0$ the space of vectors whose components lie on $\mathcal{P}_0$. Since, it is clear that the inclusion properties (a) and (b) are satisfied, we can now proceed.

Step 1. Since

$$\begin{aligned} \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} :\ \boldsymbol{v}\in\boldsymbol{V}_g, \nabla\cdot\boldsymbol{v}=0\} &= \mathsf{span}\{-1\underset{0}{\overset{0}{\square}}1,\ 0\underset{-1}{\overset{1}{\square}}0\}, \\ \{w|_{\partial K} :\ w\in W_g, \nabla w = 0\} &= \mathsf{span}\{1\underset{1}{\overset{1}{\square}}1\}, \\ M(\partial K) &= \mathsf{span}\{0\underset{0}{\overset{0}{\square}}1,\ 1\underset{0}{\overset{0}{\square}}0,\ 0\underset{0}{\overset{1}{\square}}0,\ 1\underset{0}{\overset{0}{\square}}0\} \end{aligned}$$

we have that $I_M(\boldsymbol{V}_g \times W_g) = 4-2-1 = 1$ and we can take $C_M = \mathsf{span}\{0\underset{0}{\overset{-1}{\square}}1\}$. So, we can take

$$\boldsymbol{V}_{\text{fillM}} := \mathsf{span}\{(x, -y)\}.$$

This means that

$$(\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}}) \times W_g = \mathsf{span}\{(1,0),(0,1),(x,-y)\} \times \mathsf{span}\{1\},$$

admits an $\mathcal{P}_0(\partial K)$-decomposition.

Step 2. The space constructed in this step, namely,

$$(\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}}) \times \nabla\cdot\boldsymbol{V}_g = \mathsf{span}\{(1,0),(0,1),(x,-y)\}\times\{0\},$$

does **not** admit an $\mathcal{P}_0(\partial K)$-decomposition because

$$\{w|_{\partial K} : w \in W_g, \nabla w = 0\} = \mathsf{span}\{1\} \neq \{0\} = \{w|_{\partial K} : w \in \nabla\cdot\boldsymbol{V}_g, \nabla w = 0\}.$$

Step 3. Finally, we note that $\nabla\cdot\boldsymbol{V}_g = \{0\}$ and so $I_M(\boldsymbol{V}_g \times W_g) = 1 - 0 = 1$ and $C_W = W_g$. We can then take

$$\boldsymbol{V}_{\text{fillW}} := \mathsf{span}\{(x, y)\}.$$

This means that the space

$$(\boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}} \oplus \delta\boldsymbol{V}_{\text{fillW}}) \times W_g = \mathsf{span}\{(1,0),(0,1),(x,-y),(x,y)\}\times\mathsf{span}\{1\},$$

also admits an $\mathcal{P}_0(\partial K)$-decomposition. This completes the construction.

2.5.2 Triangular and Quadrilateral Elements

Let us now consider triangular and quadrilateral elements, $M := \mathcal{P}_k(\partial K)$ and two cases of the spaces $\boldsymbol{V}_g \times W_g$. The first is only associated with rectangles, $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{Q}}_k \times \mathcal{Q}_k$; $\mathcal{Q}_k$ denotes the space of tensor product polynomials of degree at most k, and $\boldsymbol{\mathcal{Q}}_k$ denotes the space of vectors whose components lie on $\mathcal{Q}_k$. The second is $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$; $\mathcal{P}_k$ denotes the space polynomials of degree at most k, and $\boldsymbol{\mathcal{P}}_k$ denotes the space of vectors whose components lie on $\mathcal{P}_k$. The results are displayed in Table 2.4 taken from [14].

In Table 2.4, we use the notation $\mathbf{curl}\, p := (-p_y, p_x)$. We also need to define the linear function λ_i and the rational function ξ_i associated to the definition of the spaces for quadrilaterals. Let $\{\mathbf{v}_i\}_{i=1}^4$ be the set of vertices of the quadrilateral K which we take to be counter-clockwise ordered. Let $\{\mathsf{e}_i\}_{i=1}^4$ be the set of edges of K where the edge e_i connects the vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, where we set $\mathbf{v}_5 = \mathbf{v}_1$. Then, for $1 \le i \le 4$, we define λ_i to be the linear function that vanishes on edge e_i and reaches maximum value 1 in the closure of K, and ξ_i to be a rational function such that $\xi_i|_{\mathsf{e}_i} \in \mathcal{P}_1(\mathsf{e}_i)$ and $\xi_i(\mathbf{v}_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. A particular choice of ξ_i is given as follows:

$$\xi_i := \eta_{i-1}\frac{\lambda_{i-2}}{\lambda_{i-2}(\mathbf{v}_i)} + \eta_i\frac{\lambda_{i+1}}{\lambda_{i+1}(\mathbf{v}_i)}, \qquad \text{where} \qquad \eta_i := \Pi_{\substack{j=1\\ j\neq i}}^4 \frac{\lambda_j}{\lambda_j + \lambda_i}.$$

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Table 2.4 Spaces $\boldsymbol{V} \times W$ admitting an $M(\partial K)$-decomposition, where $M = \mathcal{P}_k(\partial K)$

$\boldsymbol{V}$	W	Method
K is a square and $\boldsymbol{V}_g \times W_g = \mathcal{Q}_k \times \mathcal{Q}_k$		
$\mathcal{Q}_k \oplus \mathbf{curl}\ \mathrm{span}\{x^{k+1}y, x\,y^{k+1}\} \oplus\ \mathrm{span}\{\boldsymbol{x}\,x^k y^k\}$	$\mathcal{Q}_k$	$\mathbf{TNT}_{[k]}$ [22]
$\mathcal{Q}_k \oplus \mathbf{curl}\ \mathrm{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{Q}_k$	$\mathbf{HDG}^{Q}_{[k]}$[22]
$\mathcal{Q}_k \oplus \mathbf{curl}\ \mathrm{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{Q}_k \setminus \{x^k y^k\}$	$\mathbf{BDM}_{[k]}$
K is a triangle and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$		
$\boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$	$\mathcal{P}_k$	$\mathbf{RT}_k$ [38]
$\boldsymbol{\mathcal{P}}_k$	$\mathcal{P}_k$	$\mathbf{HDG}_k$[22]
$\boldsymbol{\mathcal{P}}_k$	$\mathcal{P}_{k-1}$	$\mathbf{BDM}_k$ [4]
K is a square and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$		
$\boldsymbol{\mathcal{P}}_k \oplus \mathbf{curl}\ \mathrm{span}\{x^{k+1}y, x\,y^{k+1}\} \oplus\ \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus \mathbf{curl}\ \mathrm{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus \mathbf{curl}\ \mathrm{span}\{x^{k+1}y, x\,y^{k+1}\}$	$\mathcal{P}_{k-1}$	$\mathbf{BDM}_{[k]}$ [4]
K is a quadrilateral and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$		
$\boldsymbol{\mathcal{P}}_k \oplus_{i=1}^{ne} \mathbf{curl}\,\mathrm{span}\{\xi_4\,\lambda_3^k, \xi_4\,\lambda_4^k\} \oplus \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus_{i=1}^{ne} \mathbf{curl}\,\mathrm{span}\{\xi_4\,\lambda_3^k, \xi_4\,\lambda_4^k\}$	$\mathcal{P}_k$	(new)
$\boldsymbol{\mathcal{P}}_k \oplus_{i=1}^{ne} \mathbf{curl}\,\mathrm{span}\{\xi_4\,\lambda_3^k, \xi_4\,\lambda_4^k\}$	$\mathcal{P}_{k-1}$	(new)

The rational function η_i is constructed in such a way that its trace on ∂K is zero except on the edge $\mathbf{e}_i$, where it is equal to one.

2.5.3 General Polygonal Elements

For general polygonal elements, we have the following result.

Theorem 2.4 ([14]) *Let K be a polygonal of ne edges such that no consecutive edges lie on the same line. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$, we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = (ne-3)(\theta+1) - \frac{1}{2}\theta(\theta-1), \quad \text{and} \quad I_S(\boldsymbol{V}_g \times W_g) = k+1,$$

where $\theta := \min\{k, ne-3\}$. Moreover, we have

$$\delta \boldsymbol{V}_{\mathrm{fillM}} := \oplus_{i=1}^{ne} \mathbf{curl}\,\Psi_i,$$
$$\delta \boldsymbol{V}_{\mathrm{fillW}} := \boldsymbol{x}\,\widetilde{\mathcal{P}}_k.$$

Here

$$\Psi_i = \begin{cases} \{0\} & \textit{if}\, i = 1, 2, \\ \text{span}\{\xi_{i+1}\lambda_{i+1}^b;\ \max\{k+3-i, 0\} \le b \le k\} & \textit{if}\ 3 \le i \le ne-1, \\ \text{span}\{\xi_{i+1}\lambda_{i+1}^b;\ \max\{k+4-i, 1\} \le b \le k\} & \textit{if}\ i = ne. \end{cases}$$

The functions $\{\xi_i\}_{i=1}^{ne} \subset H^1(K)$ *are lifting functions that satisfy*

(L.1) $\xi_i|_{\mathbf{e}_j} \in \mathcal{P}_1(\mathbf{e}_j),\ \ j = 1, \ldots, ne,$
(L.2) $\xi_i(\mathbf{v}_j) = \delta_{i,j},\ \ j = 1, \ldots, ne,$

where $\delta_{i,j}$ *is the Kronecker delta.*

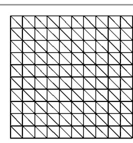
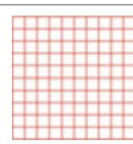
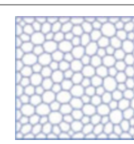
Thus results gives us an explicit, ready-to-implement description of the three spaces of our construction.

It is interesting to see how the dimension of these spaces changes when we fix the polynomial degree k and let the number of edges of the element K, ne, vary. Indeed, although the space $\delta \boldsymbol{V}_{\text{fillW}}$ remains unchanged, this is not true for $\delta \boldsymbol{V}_{\text{fillM}}$. In fact, when $k \le ne - 3$, for each additional edge in the element, the above result states that we have to add $k+1$ new basis functions to $\delta \boldsymbol{V}_{\text{fillM}}$. In particular, if $k = 1$, the dimension of $\delta \boldsymbol{V}_{\text{fillM}}$ is $2\,(ne - 3)$.

Next, we test the convergence properties of one of them. In the table below, we retake our earlier example and instead of using $\boldsymbol{V}(K) \times W(K) = \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ and $M(\partial K) = \mathcal{P}(\partial K)$ as local spaces for elements of all shapes, we consider the local spaces

$$\boldsymbol{V}(K) \times W(K) = (\boldsymbol{\mathcal{P}}_k(K) \oplus \delta \boldsymbol{V}_{\text{fillM}}) \times \mathcal{P}_k(K),$$

which, by the previous result, admit an $M(\partial K) = \mathcal{P}(\partial K)-$decomposition. We now obtain the optimal convergence order of $3 = k + 2$. This is in full agreement with our theoretical error estimates of Theorems 2.2 and 2.1, given that the approximation errors of the HDG-projection of Proposition 2.3 are both of order $k + 1$ for smooth solutions.

						
h	$\|u - u_h^\star\|_{\mathcal{T}_h}$	Rate	$\|u - u_h^\star\|_{\mathcal{T}_h}$	Rate	$\|u - u_h^\star\|_{\mathcal{T}_h}$	Rate
	$\tau = 1$					
0.1	0.15E−2	–	0.26E−2	–	0.17E−2	–
0.05	0.18E−3	3.06	0.31E−3	3.06	0.21E−3	3.02
0.025	0.23E−4	3.03	0.38E−4	3.03	0.27E−4	2.95
0.0125	0.28E−5	**3.02**	0.47E−5	**3.02**	0.35E−5	**2.96**

2.6 Extensions

We end by describing extensions of the work presented here.

Curved Elements Note that our general theory of M-decompositions for diffusion problems can be *easily* extended to curved elements by following the work done in [21].

Hanging Nodes Although in Theorem 2.4, we restricted ourselves to the case of elements with no consecutive edges in the same line, two-dimensional elements with hanging nodes can be treated by applying the general theory by simply considering that an edge with a hanging node is in fact two different edges. The three-dimensional case can be similarly treated. The case of a triangle with a hanging node is considered in [14, Section 4.2].

Local Postprocessing of the Flux By using our construction, we can locally compute *two* $H(\text{div})$-conforming approximate fluxes, see [27, Section 6.3], for the HDG approximation. This elementwise postprocessing extends the postprocessing obtained back in 2003 by Bastian and Rivière [3] (see the variations proposed, for simplicial meshes, in 2005 [17], in 2007 [31] and in 2010 in [20]). As was argued therein, see also [1, Section 2.2], $H(\text{div})$-conforming fluxes seem to be preferable to the original DG-like approximation, even if both approximations are of the same accuracy, when used on other convection-diffusion problems in which the fluxes drive the convection.

2D Versus 3D The three-dimensional case is significantly more involved than the two-dimensional case, essentially because of the computation of the space

$$\{\boldsymbol{v} \in \boldsymbol{V}_g : \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} = 0\},$$

which is very simple in 2D but very complicated in 3D. This reflects the fact that, although M-decompositions were explicitly obtained for arbitrary polygonal elements [14], in the three dimensional case, the explicit construction of M-decompositions has been done for tetrahedra, prisms, pyramids and hexahedra [15]. The *automatic* construction of M-decompositions for three-dimensional polyhedral elements of arbitrary shape constitutes the subject of ongoing research.

New Discrete H^1-Inequalities In [24], new H^1-discrete inequalities were introduced which extend to *all* spaces admitting M-decompositions similar inequalities obtained in [30, Proposition 3.2], for the well known Raviart-Thomas spaces for simplexes, and, for smaller spaces, in [7, Theorem 3.2] for the Staggered DG method.

Other Equations As pointed out in [27], this work can be extended to devise superconvergent HDG and mixed methods for the heat equation, by following [6], to the wave equation by following [12], see [26] for a Stormer-Numerov time-marching method and [39] for symplectic methods, to the velocity gradient-velocity-pressure formulation of the Stokes problem by following [10], see [25], and for methods for the the equations of linear elasticity with weakly symmetric stress approximations by following [11]. The extension to methods for the equations of linear elasticity with strongly symmetric stresses was carried out in [13]—the actual construction of the local spaces in 3D is still an open problem though. The extension to the incompressible Navier-Stokes equations was done in [24].

The theory of M-decompositions Maxwell equations constitute subject of ongoing research.

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Appendix: Proof of the Characterization of M-Decompositions

In this Appendix, we provide a proof Theorem 2.3, as it sheds light on the nature of M-decompositions. We closely follow the proof given in [27], and use the existence of the so-called *canonical* decomposition of Proposition 2.1.

Step 1. We take $\widetilde{\boldsymbol{V}} \times \widetilde{W}$ given by the canonical M-decomposition and begin by showing that

$$\dim \widetilde{\boldsymbol{V}}^{\perp} \cdot \boldsymbol{n}|_{\partial K} = \dim \widetilde{\boldsymbol{V}}^{\perp} \quad \text{and} \quad \dim \widetilde{W}^{\perp}|_{\partial K} = \dim \widetilde{W}^{\perp}.$$

Let us prove the first equality. If $\widetilde{\boldsymbol{v}}^{\perp} \in \widetilde{\boldsymbol{V}}^{\perp}$ is such that $\widetilde{\boldsymbol{v}}^{\perp} \cdot \boldsymbol{n}|_{\partial K} = 0$, for any $w \in W$, we have that

$$0 = \langle w, \widetilde{\boldsymbol{v}}^{\perp} \cdot \boldsymbol{n} \rangle_{\partial K} = (\nabla w, \widetilde{\boldsymbol{v}}^{\perp})_K + (\widetilde{w}^{\perp}, \nabla \cdot \widetilde{\boldsymbol{v}}^{\perp})_K = (\widetilde{w}^{\perp}, \nabla \cdot \widetilde{\boldsymbol{v}}^{\perp})_K$$

since $\nabla w \subset \widetilde{\boldsymbol{V}}$. Since $W \supset \nabla \cdot \boldsymbol{V}$, we can take $w := \nabla \cdot \widetilde{\boldsymbol{v}}^{\perp}$ and conclude that $\nabla \cdot \widetilde{\boldsymbol{v}}^{\perp} = 0$, which means that $\widetilde{\boldsymbol{v}}^{\perp} \in \boldsymbol{V}_{\text{sbb}}$, which means that $\widetilde{\boldsymbol{v}}^{\perp} = \boldsymbol{0}$. Thus, the first equity holds.

Now, let us prove the second equality. If $\widetilde{w}^{\perp} \in \widetilde{W}^{\perp}$ and is zero on ∂K, then, for any $\boldsymbol{v} \in \boldsymbol{V}$, we have

$$0 = \langle \widetilde{w}^{\perp}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = (\nabla \widetilde{w}^{\perp}, \boldsymbol{v})_K + (\widetilde{w}^{\perp}, \nabla \cdot \boldsymbol{v})_K = (\nabla \widetilde{w}^{\perp}, \boldsymbol{v})_K$$

since $\widetilde{W} = \nabla \cdot \boldsymbol{V}$. Since $\boldsymbol{V} \supset \nabla W$, we can now take $\boldsymbol{v} := \nabla \widetilde{w}^{\perp}$ and conclude that $\widetilde{w}^{\perp}$ is a constant on K. As a consequence $\widetilde{w}^{\perp} = 0$, and the second equality follows.

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Step 2. Next, we show that

$$\dim \operatorname{tr}(\widetilde{\boldsymbol{V}}^{\perp} \times \widetilde{W}^{\perp}) = \dim \widetilde{\boldsymbol{V}}^{\perp} \cdot \boldsymbol{n}|_{\partial K} + \dim \widetilde{W}^{\perp}|_{\partial K}.$$

To do that, we only need to show that $\widetilde{\boldsymbol{V}}^{\perp} \cdot \boldsymbol{n}|_{\partial K} \cap \widetilde{W}^{\perp}|_{\partial K} = \{0\}$. So, if $(\widetilde{\boldsymbol{v}}^{\perp}, \widetilde{w}^{\perp}) \in \widetilde{\boldsymbol{V}}^{\perp} \times \widetilde{W}^{\perp}$ we get that

$$\langle \widetilde{w}^{\perp}, \widetilde{\boldsymbol{v}}^{\perp} \cdot \boldsymbol{n} \rangle_{\partial K} = (\nabla \widetilde{w}^{\perp}, \widetilde{\boldsymbol{v}}^{\perp})_K + (\widetilde{w}^{\perp}, \nabla \cdot \widetilde{\boldsymbol{v}}^{\perp})_K = 0,$$

because $\nabla \widetilde{w}^{\perp} \in \nabla W \subset \widetilde{\boldsymbol{V}}$ and because $\nabla \cdot \widetilde{\boldsymbol{v}}^{\perp} \in \nabla \cdot \widetilde{\boldsymbol{V}} \subset \widetilde{W}$.

Step 3. By the inclusion property (a), the number

$$\begin{aligned} I :=\ & \dim M - \dim \widetilde{\boldsymbol{V}}^{\perp} - \dim \widetilde{W}^{\perp} \\ =\ & \dim M - \dim \widetilde{\boldsymbol{V}}^{\perp} \cdot \boldsymbol{n}|_{\partial K} - \dim \widetilde{W}^{\perp}|_{\partial K}. \end{aligned}$$

is always nonnegative and is equal to zero if and only if property (c) holds. Next, we show that $I = I_M(\boldsymbol{V} \times W)$; this is the *key* computation of the proof. Indeed, we have

$$\begin{aligned} I :=& \dim M - \dim \widetilde{\boldsymbol{V}}^{\perp} - \dim \widetilde{W}^{\perp} \\ =& \dim M - (\dim \boldsymbol{V} - \dim \widetilde{\boldsymbol{V}}) - (\dim W - \dim \widetilde{W}) \\ =& \dim M - (\dim \boldsymbol{V} - \dim \nabla W - \dim \boldsymbol{V}_{\text{sbb}}) - (\dim W - \dim \nabla \cdot \boldsymbol{V}) \\ =& \dim M - (\dim \boldsymbol{V} - \dim \nabla \cdot \boldsymbol{V} - \dim \boldsymbol{V}_{\text{sbb}}) - (\dim W - \dim \nabla W) \\ =& \dim M - (\dim\{\boldsymbol{v} \in \boldsymbol{V} : \nabla \cdot \boldsymbol{v} = 0\} - \dim\{\boldsymbol{v} \in \boldsymbol{V} : \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} = 0\}) \\ & \quad - \dim\{w \in W : \nabla w = 0\} \\ =& \dim M - \dim\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} - \dim\{w|_{\partial K} : w \in W, \nabla w = 0\} \\ =& I_M(\boldsymbol{V} \times W). \end{aligned}$$

Step 4. Now, by the inclusion property (a), we have that

$$\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}, \nabla \cdot \boldsymbol{v} = 0\} \oplus \{w|_{\partial K} : w \in W, \nabla w = 0\} \subset M,$$

where the sum is $L^2(\partial K)$-orthogonal since

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, w \rangle_{\partial K} = (\nabla \cdot \boldsymbol{v}, w)_K + (\boldsymbol{v}, \nabla w)_K = 0$$

if $\nabla \cdot \boldsymbol{v} = 0$ and $\nabla w = 0$. Finally, since the M-index $I_M(\boldsymbol{V} \times W)$ is zero by property (c), the equality holds. This completes the proof of the characterization Theorem 2.3.

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