An interest rate cap ensures that a floating rate note does not exceed a prescribed value over a prescribed time. The prescribed value is the 'cap rate'.

**Mechanics** The floating rate note resets the reference Libor amount periodically (say quarterly). The time between resets is the tenor.

**Rough Example** Consider an interest rate cap with a five year life, cap rate of 4%, and quarterly tenor.

Suppose on a reset date Libor is 5%. The floating rate note would require a payment of

$$1,000,000 \times 0.05 \times 0.25 = 125,000$$

While the cap would reduce it to 100,000.

The cap, then, is worth $25k ... in 3 months.

**AN INTEREST RATE CAP IS A PORTFOLIO OF INTEREST RATE OPTIONS**

Consider a cap with a life, \( T \), against principal, \( L \), and cap rate \( R \).
Let the reset dates be given by \( t_k \), \( k = 1, \ldots, n \), and let \( t_{n+1} = T \).

Let \( R_{t_k} \) be the realized Libor rate on \( [t_k, t_{k+1}] \) (observed on \( t_k \)).

At time \( t_{n+1} \), the cap gives payoff

\[
L \cdot (t_{n+1} - t_k) \cdot \max \left( R_{t_k} - R, 0 \right)
\]

* \( R \) and \( R \) expressed expressed in same compounding frequency as resets.

Each such call option above is called a caplet. It is an option on a future rate to be observed in the future, paid at a date later than that of observation.

An interest cap is simply a portfolio of caplets.

A caplet can also be defined as a put on a zero-coupon bond.

This can be seen by rewriting \( C_1 \) by beginning by writing the payoff of \( C_1 \) discounting back

\[
6 \text{ to } t_k: \quad \frac{L \cdot (t_{n+1} - t_k)}{1 + R_{t_k} \cdot (t_{n+1} - t_k)} \cdot \max \left( R_{t_k} - R, 0 \right)
\]
This is equivalent to, letting $\Delta b_k = b_{k+1} - b_k$,

$$\max \left( \frac{R_{b_k} - R}{L \Delta b_k + 1 + R_{b_k} \Delta b_k}, 0 \right)$$

$$= \max \left( \frac{L \Delta b_k R_{b_k} - L \Delta b_k R}{1 + R_{b_k} \Delta b_k}, 0 \right)$$

$$= \max \left( \frac{L + L \Delta b_k R_{b_k} - L - L \Delta b_k R}{1 + R_{b_k} \Delta b_k}, 0 \right)$$

$$= \max \left( \frac{L \frac{1 + \Delta b_k R_{b_k}}{1 + \Delta b_k R_{b_k}} - L \frac{1 - \Delta b_k R_{b_k}}{1 + \Delta b_k R_{b_k}}}{1 + R_{b_k} \Delta b_k}, 0 \right)$$

$$= \max \left( L - \frac{L \frac{1 - \Delta b_k R_{b_k}}{1 + \Delta b_k R_{b_k}}}{1 + R_{b_k} \Delta b_k}, 0 \right)$$

At time $b_k$,

$$L \frac{1 - R \Delta b_k}{1 + R_{b_k} \Delta b_k}$$

is the value of a zero coupon bond paying $L \left( 1 - R \Delta b_k \right)$ at time $b_{k+1}$.

(02) This can be seen as a put on a zero coupon bond expiring at time $b_k$. The face is of course $L \left( 1 - R \Delta b_k \right)$ maturing at time $b_{k+1}$.

We know how to price it! <<(on)> Jump to page 11 and return
Floors

An interest rate floor ensures the interest rate of a floating rate note does not fall below a prescribed value over a prescribed time. The prescribed value is the floor rate. It pays when interest rates fall below the floor rate.

Using the same notation as before, an floor is a portfolio of floorlets

\[
L \Delta t \max(R - R_k, 0)
\]

or

\[
\max\left(\frac{L (1 - R_k \Delta t)}{1 + R_k \Delta t} - L, 0\right)
\]

That is, a floor is a portfolio of zero coupon bonds.

Collar

A collar ensures rates are between two specified values. It is a long position in a cap and a short position in a floor.

Swaps, Caps, and Floors

For a swap agreement to receive Libor and pay \( R \), we have, for floors and caps with strike \( R \), that a collar has the following payoff.

\[
\begin{cases} 
R_k & \text{if Libor} > R \\
R_k - R & \text{if Libor} < R \\
\end{cases}
\]

(Cap pays)

\[
\text{Floor pays} \quad -(R - R_k) = R_k - R
\]

In any case, the payoff is \( R_k - R \).
That is, pay fixed, receive LIBOR. That is, a collar is equivalent to a swap or:

\[ \text{Value of Swap} = \text{Value of Cap} - \text{Value of Floor} \]

**Valuation of Each Caplet** (as an option on rates)

Each caplet has payoff at time \( t_{k+1} \) for rate observed at \( t_k \) of

\[ L \Delta t_k \cdot \max (R_{t_k} - R, 0) \]

Using Black's Model the value of \( t_{k+1} \) caplet today is

\[ L \Delta t_k \cdot P(0, t_{k+1}) \cdot \left[ F_{t_k} \cdot N(d_1) - R \cdot N(d_2) \right] \]

with

\[ d_1 = \frac{\ln \left( \frac{F_{t_k}}{R} \right) - \frac{1}{2} \sigma_{t_k}^2 \Delta t_k}{\sigma_{t_k} \sqrt{\Delta t_k}} \]

\[ d_2 = d_1 - \sigma_{t_k} \sqrt{\Delta t_k} \]

with

\[ F_{t_k} = \text{forward interest rate from } t_k \text{ to } t_{k+1} \text{ observed today} \]

\[ \sigma_{t_k} = \text{the volatility of } F_{t_k} \]

The discount factor reflects that payment is made at \( t_{k+1} \).
Whereas the volatility is scaled by $\sqrt{T_k}$ since the rate $R_{t_k}$ is observed at $t_k$.

NB: Each caplet and floorlet is valued separately here, introducing the possible need for a forward structure in rate $R_{t_k}$ volatility.

**Example**

Consider a contract that caps LIBOR at 8%annually (compounded quarterly) for 3 months beginning in one year [i.e. a caplet].

Suppose the LIBOR zero curve is flat at 7% annually (compounded quarterly).

Let the 3 months forward rate for this caplet have a volatility of 20% annually.

We need, to use Black's model: $F_k$, $R$, $\delta$, $L$, $\Delta t_k$, $\Delta t_{k+1}$, $P(0, t_{k+1})$

We have

$F_k := 0.07$

$R := 0.08$

$\delta := 0.20$

$L := 10$ (MM)

$\Delta t_k := 1.25$ (t.k := 1)

$\Delta t_{k+1} := 0.25$

$P(0, t_{k+1}) = e^{-t_{k+1}}$

$e^{\frac{-r}{\Delta t_{k+1}}} = \left(1 + 0.07 \cdot 0.25\right)$

$r = 6.94%$
Putting it all together, we have the value of the caplet is

\[ 10 \cdot 0.25 \cdot e^{-0.044 \cdot 1.25} \cdot \left[ 0.07 N(d_1) - 0.08 N(d_2) \right] \]

with

\[ d_1 = \frac{\ln \left( \frac{0.07}{0.08} \right) + \frac{1}{2} 0.02^2 \cdot 1}{0.2 \cdot 1} \approx -0.5677 \]

This gives a value of $5162.

NB: Caps and caplets can be quoted in vol.

**On Volatilities**

Spot Volatilities: Volatility for Caplet

Flat Volatilities: Volatility for Cap

\[ e^{-t} \]

Maturity

**Justification/ Back Calculation of Numeraires**

Consider a world that is forward risk neutral with respect to a zero coupon bond maturing at \( t + \tau \).

The price of a caplet is
\[ P(0, t_{n+1}) \Phi_{t_{n+1}} \left( L \cdot (E_{t_{n+1}} - E_k) \cdot \max (R_{t_{n+1}} - R, 0) \right) \]

with \( \Phi_{t_{n+1}}(\cdot) \) the expected value in the world forward risk neutral w.r.t. a zero coupon bond maturing at \( t_{n+1} \).

It simplifies to

\[ L \cdot (E_{t_{n+1}} - E_k) \cdot P(0, t_{n+1}) \Phi_{t_{n+1}} \left( \max (R_{t_{n+1}} - R, 0) \right) \]

If the forward rate, \( R_{t_{n+1}} \), is assumed to have constant volatility, it is log normal in this world with standard deviation \( \sqrt{\ln(R_{t_{n+1}})} \) equal to \( \sigma \cdot \sqrt{t_{n+1}} \).

As before, this gives

\[ L \cdot (E_{t_{n+1}} - E_k) \cdot P(0, t_{n+1}) \cdot \Phi_{t_{n+1}} \left( R_{t_{n+1}} N(d_1) \right) \]

with \( d_1 \) and \( d_2 \) as usual, and

\[ \Phi_{t_{n+1}}(R_k) = F_k \]