1. Hull 31.28

2. Assume that under the risk neutral measure that zero coupon bonds follow
\[ dP(t, T) = r(t)P(t, T)dt + v(t, T)P(t, T)dZ. \]

(a) Show that \( d\ln P(t, T) = (r(t) - \frac{1}{2}v(t, T)^2) dt + v(t, T) dZ \)

(b) For \( f(t, T_1, T_2) \) that the forward rate between \( T_1 \) and \( T_2 \) as seen at time \( t \), determine \( df \).

(c) For \( R(t, T) \) the zero rate, why is \( R(t, T) = f(0, t, T) + \int_0^t df(\tau, t, T) \)?

(d) (For this problem, Hull’s Technical Note 31 will be a ton of help, but you should try it first without.) The Hull-White single factor model is given by
\[ dr = (\theta(t) - a \cdot r) dt + \sigma dZ \]
and under these dynamics,
\[ P(t, T) = A(t, T)e^{-B(t, T)\cdot r(t)} \]
with \( B(t, T) = \frac{1-e^{-a(T-t)}}{a} \).

i. Prove \( v(t, T) = \sigma \frac{1-e^{-a(T-t)}}{a} \)?

ii. In class we showed that
\[ r(t) = F(0, t) + \int_0^t v(\tau, t)v_t(\tau, t)d\tau - \int_0^t v_t(\tau, t) dZ. \]
What is \( r(t) \) given the form of \( v(t, T) \) above?

iii. Determine \( \theta(t) \) for the Hull-White model.

iv. What is the volatility of a forward bond under Hull-White? You may use any results proven in lecture.

v. Explain how you would price a European option with strike \( K \) and expiry \( s \) on a zero-coupon bond with maturity \( T \) at time \( t = 0 \) under the Hull-White model. What model parameters would you need? What market prices would you need? Be explicit.