Portfolio Statistics and Optimization

Introduction

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The Need for a Framework

Every solution requires a formulation.
Every anomaly requires a reference model by which it may be measured.

Consider

An Apparent Excess Return

<table>
<thead>
<tr>
<th>Value of $1 Invested</th>
<th>Date</th>
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Motivation for Excess and Risk

In the preceding figure, the blue line shows the growth of a dollar investment in a market index.

The red line clearly outperforms ... eventually.

But even a cursory examination shows this second asset has more risk than the market.

- It tempts ruin in early 2009.
- Overall, it seems more ‘volatile’.
- In what sense does it really outperform the market?

Is a bet that pays $100 1% of the time and zero all others a better bet than one that pays $0.90 surely?
In the present example, the red line asset is simply the market leveraged $2\times$.

- If we subtract off $2\times$ the market, there’s no longer any excess.
- Our intuition of risk tells us that this asset is at least $2\times$ as ‘risky’.

A seminal contribution to understanding just such a problem was the Capital Asset Pricing Model (CAPM)

$$r_t - r_f = \beta (m_t - r_f) + \epsilon_t$$

Matching our intuition, the math gives $\beta = 2$, and $\epsilon_t \equiv 0$ for the red line asset.
Would that the world was so simple, but the problems that arise from this simple and direct model are also motivators for understanding mathematical finance generally:

- A given stock’s $\beta$ may change over time, a signal that time increments are not all treated equally in equity returns.
- We shall see that this model finds justification in a normal distribution of equity returns, but this assumption is not reflected in realized historical returns.

In the main, the model is prescriptive rather than empirical; it is no less instructive, however.
The motivation for the current text is to understand portfolio statistics and optimization, largely through the lens of standard models like CAPM and modern portfolio theory, illustrated succinctly in the now standard mean-variance problem

$$\min_w w' \Sigma w$$

$$\mu' w = \mu^*$$

$$1' w = 1.$$ 

Immediately central are statistics such as the mean and covariance
Our Path: Statistics

- Common univariate and multivariate distributions
- Expectation, estimators, bias, and variance
- Covariance, correlation, eigenvalues, and independence
- Copulas
- Maximum likelihood
- Ordinary least squares
  - Projections
  - Parameter distributions and confidence intervals
  - Hypothesis testing
  - Variable selection
- Generalized least squares
Our Path: Optimization

- Multivariate calculus; esp. Taylor series
- Convex functions
- Unconstrained optimization
- Constrained optimization
- Linear and quadratic programming problems
- Mean-variance optimization
- Coherent measures of risk as optimization problems
fin.