The integrating factor method for solving partial differential equations may be used to solve linear, first order differential equations of the form:

\[
\frac{dy}{dx} + a(x)y = b(x),
\]

where \(a(x)\) and \(b(x)\) are continuous functions. We will say that an equation written in the above way is written in the standard form. The method for solving linear, first order differential equations using the integrating factor method may be broken down into the following steps.

1. Write the differential equation in the standard form: \(\frac{dy}{dx} + a(x)y = b(x)\).
2. Determine the integrating factor.
   
   \[
   \text{Integrating Factor} = e^{\int a(x)dx}
   \]
3. Multiply the equation in standard form by the integrating factor.
   
   \[
   e^{\int a(x)dx} \left( \frac{dy}{dx} + a(x)y \right) = b(x)e^{\int a(x)dx}
   \]
4. Using the product and chain rule of differentiation, write the left hand side of the equation in the following way:
   
   \[
   e^{\int a(x)dx} \left( \frac{dy}{dx} + a(x)y \right) = \frac{d}{dx} \left( e^{\int a(x)dx}y \right)
   \]
   
   So,
   
   \[
   \frac{d}{dx} \left( e^{\int a(x)dx}y \right) = b(x)e^{\int a(x)dx}
   \]
5. Integrate both sides of the new equation:
   
   \[
   \int \frac{d}{dx} \left( e^{\int a(x)dx}y \right) dx = \int b(x)e^{\int a(x)dx} dx.
   \]
   
   The Fundamental Theorem of Calculus shows that
   
   \[
   \int \frac{d}{dx} \left( e^{\int a(x)dx}y \right) dx = e^{\int a(x)dx}y + C_1,
   \]
   
   where \(C_1\) is an arbitrary constant. You must solve the other integral. Call its solution \(B(x) + C_2\), where \(C_2\) is a constant due to the integration. So,
   
   \[
   e^{\int a(x)dx}y = B(x) + C_3.
   \]
   
   I grouped the constants together and called the result \(C_3\).
6. Divide by the integrating factor to get the solution:
   
   \[
   y = B(x)e^{-\int a(x)dx} + C_3e^{-\int a(x)dx}.
   \]

For the fourth step, you must remember the product and chain rule of differentiation as well as the Second Fundamental Theorem of Calculus. The Second Fundamental Theorem of Calculus states that for a continuous function \(f(x)\),

\[
\frac{d}{dx} \int f(x) dx = f(x).
\]
Let us verify step 4:

\[
\frac{d}{dx} \left( e^{\int a(x) dx} y \right) = e^{\int a(x) dx} \frac{dy}{dx} + \left( \frac{d}{dx} \int a(x) dx \right) e^{\int a(x) dx} y
\]

\[
= e^{\int a(x) dx} \frac{dy}{dx} + a(x) e^{\int a(x) dx} y
\]

\[
= e^{\int a(x) dx} \left( \frac{dy}{dx} + a(x) y \right).
\]

To make this check a little less abstract, I will show that \( \frac{d}{dx}(e^{4x}y) = e^{4x}(y' + 4y) \). We will use this in the first example.

\[
\frac{d}{dx}(e^{4x}y) = e^{4x} \frac{dy}{dx} + (4e^{4x}) y
\]

\[
= e^{4x} \left( \frac{dy}{dx} + 4y \right).
\]

Now, we will use the integrating factor method to solve the first example. Find the solution of \( y' = -4y \).

1. Write the equation in standard form.

\[
y' + 4y = 0
\]

2. Find the integrating factor.

\[
e^{\int 4 dx} = e^{4x}
\]

3. Multiply by the integrating factor.

\[
e^{4x}(y' + 4y) = 0
\]

4. Rewrite the left hand side of the equation.

\[
\frac{d}{dx}(e^{4x}y) = 0
\]

5. Integrate.

\[
e^{4x}y = C
\]

6. Divide by the integrating factor to get the solution.

\[
y = Ce^{-4x}
\]

Let us do another example. Find the solution of \( y' - 2y = x \).

1. Write the equation in standard form.

\[
y' - 2y = x
\]

2. Find the integrating factor.

\[
e^{\int -2 dx} = e^{-2x}
\]

3. Multiply by the integrating factor.

\[
e^{-2x}(y' - 2y) = x e^{-2x}
\]

4. Rewrite the left hand side of the equation.

\[
\frac{d}{dx}(e^{-2x}y) = x e^{-2x}
\]

5. Integrate. This will require integration by parts.

\[
e^{-2x}y = -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C_3
\]

6. Divide by the integrating factor to get the solution.

\[
y = -\frac{1}{2} x - \frac{1}{4} + C_3 e^{2x}.
\]
Let us do a third example. Find the solution to \( \frac{1}{x} \frac{dy}{dx} - \frac{2}{x^2} y = x \cos x \) for \( x > 0 \).

1. Write the equation in standard form.
   \[ y' - \frac{2}{x} y = x^2 \cos x \]

2. Find the integrating factor.
   \[ e^{\int \left(-\frac{2}{x}\right) dx} = e^{-2 \ln |x|} = x^{-2} \]

3. Multiply by the integrating factor.
   \[ x^{-2}(y' - \frac{2}{x} y) = \cos x \]

4. Rewrite the left hand side of the equation.
   \[ \frac{d}{dx}(x^{-2} y) = \cos x \]

5. Integrate.
   \[ x^{-2} y = \sin x + C_1 \]

6. Divide by the integrating factor to get the solution.
   \[ y = x^2 \sin x + C_1 x^2. \]