Problem. Solve the discrete initial value problem $y_{n+1} = 4y_n + 4^n, y_0 = 1$.

Solution. We may solve this problem by finding the homogeneous and particular solutions to the difference equation and then using the initial condition to determine the unknown constant in the homogeneous solution. So, let us first find the homogeneous solution. The homogeneous solution to the difference equation is a solution to

$$y^{(h)}_{n+1} = 4y^{(h)}_n,$$

where the $(h)$ superscript is merely to remind us that we are dealing with the homogeneous problem. In order to find the solution, we guess $y^{(h)}_n = r^n$. Substituting this guess into the homogeneous equation yields

$$r^{n+1} = 4r^n \Rightarrow r^n(r - 4) = 0 \Rightarrow r = 4.$$

Hence, our homogeneous solution is $y^{(h)}_n = C_14^n$, where $C_1$ is an arbitrary constant. Now, we must find the particular solution. Usually, when the inhomogeneous term is of the form $t^n$ where $t$ is a constant, we guess a particular solution of the form $C_p t^n$. However, we see that in this instance such a guess is the same as the homogeneous solution. So, we multiply our guess for the particular solution by the lowest power of $n$ such that the guess no longer coincides with the homogeneous solution. In this case, we multiply the guess by $n$. So, our guess of the particular solution is $y^{(p)}_n = C_p 4^n n$. Substituting this guess into the original difference equation yields to solve for $p$, we get that

$$C_p 4^{n+1}(n + 1) = 4C_p 4^n n + 4^n \Rightarrow 4C_p(n + 1) = 4C_p n + 1 \Rightarrow 4C_p = 1 \Rightarrow C_p = \frac{1}{4}.$$  

The particular solution is $y^{(p)}_n = 4^{n-1} n$. The general solution to the difference equation is the sum of the homogeneous and particular solutions, so $y_n = C_14^n + 4^{n-1} n$. Recall that $y_0 = 1$, so $y_0 = 1 = C_1$. The solution to the problem with the given initial condition is

$$y_n = 4^n + n4^{n-1}.$$
Problem. Solve the system

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for a general initial condition.

Solution. The solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{tA} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix},$$

where

$$A = \begin{bmatrix} 7 & -3 \\ 2 & 2 \end{bmatrix}.$$

To finish the problem, we merely need to find $e^{tA}$. When the matrix $A$ is diagonalizable (that is, we can write $A = PDP^{-1}$ as we have seen in class), we can find the exponential easily. Observe that

$$\begin{bmatrix} 7 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}.$$

So,

$$\begin{bmatrix} 7 & -3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

This implies that

$$e^{tA} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3e^{5t} - 2e^{4t} & -3e^{5t} + 3e^{4t} \\ 2e^{5t} - 2e^{4t} & -2e^{5t} + 3e^{4t} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3e^{5t} - 2e^{4t} & -3e^{5t} + 3e^{4t} \\ 2e^{5t} - 2e^{4t} & -2e^{5t} + 3e^{4t} \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}.$$