Solutions to Supplementary Problems for Week 5

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Salt Tank Problem. A large tank initially contains 240 gallons of water and 35 pounds of salt. At time \( t = 0 \), brine begins to flow into the tank at a rate of 4 gallons per minute. The brine contains salt at a concentration of 4 pounds per gallon. Well-mixed solution is allowed to drain from the tank at a rate of 7 gallons per minute. Find the amount of salt in the tank at time \( t \) (until the tank runs dry).

Solution. Let \( S(t) \) denote the pounds of salt in the tank at time \( t \). We are given that the tank initially contains 35 pounds of salt. This initial condition may be written as \( S(0) = 35 \). We will need this initial condition later to solve for the unknown coefficient we will have in our equation for \( S(t) \). Now, in order to calculate the amount of salt in the tank at later times, we must take into account the flow of salt in and out of the tank. This flow will determine the rate of change for the amount of salt in the tank, which is the derivative of \( S(t) \) with respect to the time \( t \). So,

\[
\frac{dS}{dt} = \text{(Flow of salt in)} - \text{(Flow of salt out)}.
\]

We know that brine flows into the tank at a rate of 4 gallons per minute. The brine contains salt at a concentration of 4 pounds per gallon. To determine how many pounds of salt are entering the tank as a result of this flow, we multiply the concentration of the salt in the brine with the amount of brine flowing in per minute:

\[
\text{(Flow of salt in)} = \left( \frac{4 \text{ pounds of salt}}{1 \text{ gallon of brine}} \right) \left( \frac{4 \text{ gallons of brine}}{1 \text{ minute}} \right) = \frac{16 \text{ pounds of salt}}{\text{minute}}.
\]

For the flow of salt out of the tank, we know the amount of solution flowing out, but we do not know the concentration of the salt in the flow. We can calculate the concentration of the salt in the fluid flowing out of the tank by dividing the total amount of salt in the tank by the number of gallons of solution in the tank. The important thing to note here is that the number of gallons of water in the tank varies. We are letting 4 gallons flow into the tank per minute, but we are draining the tank at a rate of 7 gallons per minute. This means that the overall water supply in the tank is decreasing at a rate of 3 gallons per minute. The total amount of water in the tank at time \( t \) is \( 240 - 3t \). So,

\[
\text{(Flow of salt out)} = \left( \frac{S(t) \text{ pounds of salt}}{240 - 3t \text{ gallons}} \right) \left( \frac{7 \text{ gallons}}{1 \text{ minute}} \right) = \frac{7S(t)}{240 - 3t} \text{ pounds of salt per minute}.
\]
Therefore,
\[
\frac{dS}{dt} = 16 - \frac{7S}{240 - 3t}.
\]
We now have a first order differential equation that we can solve to find \(S\) at all times \(t\). We will solve this differential equation using the integrating factor method. The first step in the integrating factor method is to write the differential equation in standard form \((S' + a(t)S = b(t))\):
\[
\frac{dS}{dt} + \frac{7}{240 - 3t}S = 16.
\]
Second, find the integrating factor. The integrating factor is
\[
\exp\left(\int \frac{7}{240 - 3t} dt\right) = \exp\left(\frac{7}{3} \int \frac{1}{80 - t} dt\right) = \exp\left(-\frac{7}{3} \ln |80 - t|\right) = |80 - t|^{-7/3}.
\]
Note that \(\exp(x) = e^x\). This new notation is used just to improve the readability of the above equation. I will drop the absolute value from the integrating factor as we will not be letting \(t > 80\). At \(t = 80\), the tank is empty. Next, we multiply both sides of our differential equation in standard form by this integrating factor to get
\[
(80 - t)^{-7/3} \frac{dS}{dt} + \frac{7}{240 - 3t} S = 16(80 - t)^{-7/3}.
\]
Using the product rule of differentiation, we can write the above equation as
\[
\frac{d}{dt} ((80 - t)^{-7/3} S) = 16(80 - t)^{-7/3}.
\]
Integrate both sides to get
\[
(80 - t)^{-7/3} S = \int 16(80 - t)^{-7/3} dt = \frac{12}{(80 - t)^{4/3}} + C_1,
\]
where \(C_1\) is an arbitrary constant. The above integral may be found using the standard \(u\) substitution. Finally, divide both sides by \((80 - t)^{-7/3}\) to get an equation for \(S(t)\):
\[
S(t) = 12(80 - t) + C_1(80 - t)^{7/3}.
\]
As was mentioned earlier, we will use our initial condition to find \(C_1\). The initial condition is \(S(0) = 35\). So,
\[
36 = 12(80) + C_1(80)^{7/3} \Rightarrow -924 = C_1(80^{7/3}) \Rightarrow C_1 = -924(80^{-7/3}) \approx -0.034.
\]
Our solution to this initial value problem is
\[
S(t) = 12(80 - t) - 924(80^{-7/3})(80 - t)^{7/3}.
\]
**Coroner’s Report.** A body is discovered in a park at 3 am in the morning. All through the evening the temperature is steady at 45°F Fahrenheit. The temperature of the body is measured at 3 am and again at 4 am just before moving the body to the morgue. The temperatures are 80° and 60° Fahrenheit, respectively. Estimate the time of death using Newton’s law of heating/cooling. Start with the differential equation.

**Solution.** Let \( u(t) \) denote the temperature of the body. Let \( f(t) \) denote the temperature of the environment. We are given the temperature of the body at two different times. I will arbitrarily set 3 am to be \( t = 0 \). We are measuring \( t \) in hours, so 4 am will be \( t = 1 \). We may write the two given conditions for the body temperature as \( u(0) = 80 \) and \( u(1) = 60 \). The time of death will then occur for some negative \( t \). Newton’s law of cooling asserts that

\[
\frac{du}{dt} = k(f - u), \; k > 0.
\]

We are given that \( f(t) = 45 \). So,

\[
\frac{du}{dt} = k(45 - u) = 45k - ku.
\]

The above differential equation is a first order, linear differential equation that we may solve using the integrating factor method. The first step in the integrating factor method is to write the equation in standard form \((u' + a(t)u = b(t))\):

\[
\frac{du}{dt} + ku = 45k.
\]

Next, find the integrating factor. The integrating factor is

\[
e^{\int k dt} = e^{kt}.
\]

Multiply the differential equation in standard form by the integrating factor:

\[
e^{kt}\frac{du}{dt} + ke^{kt}u = 45ke^{kt}.
\]

We may rewrite the above differential equation using the product rule of differentiation. We now have that

\[
\frac{d}{dt} (e^{kt}u) = 45ke^{kt}.
\]

Integrate the above equation with respect to \( t \) to get

\[
e^{kt}u = 45e^{kt} + C_1,
\]

where \( C_1 \) is an arbitrary constant. Divide both sides by \( e^{kt} \) to find the general solution for the temperature of the body. The general solution is

\[
u(t) = 45 + C_1e^{-kt}.
\]
We have two unknown constants in the above equation, namely \( C_1 \) and \( k \). We will use our two initial conditions to find them. If \( u(0) = 80 \), then
\[
80 = 45 + C_1 e^0 \Rightarrow 35 = C_1.
\]
If \( u(1) = 60 \), then
\[
60 = 45 + 35 e^{-k} \Rightarrow \frac{15}{35} = e^{-k} \Rightarrow -k = \ln \left( \frac{15}{35} \right) \Rightarrow k = \ln \left( \frac{7}{3} \right).
\]
The solution becomes
\[
u(t) = 45 + 35 \left( \frac{3}{7} \right)^t.
\]
To determine the time of death, assume that the temperature of the body of the victim was 98.6° Fahrenheit at the time of death. Then,
\[
98.6 = 45 + 35 \left( \frac{3}{7} \right)^{t_{t.o.d.}}.
\]
Solving for \( t_{t.o.d.} \), we find that
\[
t_{t.o.d.} = \frac{\ln(53.6/35)}{\ln(3/7)} \approx -0.50.
\]
As \( t \) is measured in hours, \( t_{t.o.d.} \approx -0.5 \) implies that the time of death was approximately half an hour before \( t = 0 \) or 3 a.m. The time of death was approximately 2:30 a.m. Note that the time of death is dependent upon your initial assumption for the temperature of the victim's body. A different assumption in that temperature will lead to a slightly different estimate for the time of death.

**Logistic Equation.** Find the general solution of the differential equation
\[
\frac{dy}{dt} = 3y(7 - y).
\]
Then find the three particular solutions satisfying the initial conditions \( y(0) = 2 \), \( y(0) = 8 \), and \( y(0) = -2 \), respectively.

**Solution.** In order to find the general solution to the logistic equation, we will use the separation of variables technique. Separating the \( y \) and \( t \) variables leads to
\[
\frac{dy}{y(7 - y)} = 3dt.
\]
We would like to integrate the above equation to solve for \( y \); however, the term on the left hand side requires a simplification before we may do so. The left hand side may be simplified using a partial fraction expansion. Let
\[
\frac{1}{y(7 - y)} = \frac{A}{y} + \frac{B}{7 - y}.
\]
In order to determine \( A \) and \( B \), multiply both sides of the above equation by \( y(7 - y) \):

\[
1 = A(7 - y) + B(y).
\]

The fastest way to determine \( A \) is to set \( y = 0 \). Then, we have that

\[
1 = 7A \Rightarrow A = \frac{1}{7}.
\]

Similarly, we may determine \( B \) by setting \( y = 7 \):

\[
1 = 7B \Rightarrow B = \frac{1}{7}.
\]

Therefore,

\[
\frac{1}{y(7 - y)} = \frac{1}{7y} + \frac{1}{7 - y}.
\]

Our separated differential equation now reads

\[
\left[ \frac{1}{y} + \frac{1}{7 - y} \right] dy = 3 dt.
\]

Multiplying both sides by 7 to eliminate the fractions yields

\[
\left[ \frac{1}{y} + \frac{1}{7 - y} \right] dy = 21 dt.
\]

Now, we may integrate both sides. The result is

\[
\ln |y| - \ln |7 - y| = 21t + C_1,
\]

where \( C_1 \) is an arbitrary constant. Remember that \( \ln(A) - \ln(B) = \ln(A/B) \). Hence,

\[
\ln \left| \frac{y}{7 - y} \right| = 21t + C_1.
\]

Exponentiating both sides gives us that

\[
\left| \frac{y}{7 - y} \right| = e^{21t+C_1} = e^{C_1}e^{21t} = C_2e^{21t},
\]

where \( C_2 \) is a positive constant. Removing the absolute values will allow \( C_2 \) to be an arbitrary nonzero constant which I will call \( C_3 \):

\[
\frac{y}{7 - y} = C_3e^{21t}.
\]
To find the general solution to the differential equation, we now just solve for \( y \):

\[
\frac{y}{7 - y} = C_3 e^{21t}
\]

\[
y = (7 - y)C_3 e^{21t}
\]

\[
y(1 + C_3 e^{21t}) = 7C_3 e^{21t}
\]

\[
y = \frac{7C_3 e^{21t}}{1 + C_3 e^{21t}}.
\]

Our general solution to this logistic equation is

\[
y = \frac{7C_3 e^{21t}}{1 + C_3 e^{21t}}.
\]

To finish the problem, we must find \( C_3 \) for the three different initial conditions. One easy way to find the constant for the three cases is to use the equation

\[
\frac{y}{7 - y} = C_3 e^{21t}
\]

from a few lines up rather than our general solution. If \( y(0) = 2 \), then

\[
\frac{2}{7 - 2} = C_3 e^{21(0)} \Rightarrow C_3 = \frac{2}{5}.
\]

If \( y(0) = 8 \), then

\[
\frac{2}{7 - 8} = C_3 e^{21(0)} \Rightarrow C_3 = -2.
\]

If \( y(0) = -2 \), then

\[
\frac{2}{7 - (-2)} = C_3 e^{21(0)} \Rightarrow C_3 = -\frac{2}{9}.
\]

The problem is complete.

**Radioactive Decay.** Utopium (recall from the lecture) has a half life of 7 yrs. Utopium decays to Hopium which has a half-life of 14 yrs. Starting from 1 pound of pure Utopium at time \( t = 0 \), find the amount of Utopium and Hopium at each subsequent time \( t \).

**Solution.** For this problem, let \( U(t) \) denote the amount of Utopium at time \( t \). Let \( H(t) \) denote the amount of Hopium at time \( t \). We will first consider the decay of Utopium. We are given that we have 1 pound of Utopium at \( t = 0 \). This leads to the initial condition \( U(0) = 1 \). Since the only change in Utopium is due to radioactive decay, we use the standard radioactive decay differential equation to describe the change in Utopium per year:

\[
\frac{dU}{dt} = -k_U U,
\]

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where \( k_U \) is some positive constant related to the half-life of Utopium. Specifically, \( k_U = \frac{\ln(2)}{t_{1/2,U}} = \frac{\ln(2)}{7} \) where \( t_{1/2,U} \) is the half-life of Utopium. We may solve for \( U(t) \) using the separation of variables method or the integrating factor method. The solution of this differential equation is

\[
U = U(0) \left( \frac{1}{2} \right)^{t/7} = \left( \frac{1}{2} \right)^{t/7} = e^{-k_U t}.
\]

In order to derive an equation that describes the amount of Hopium at time \( t \), we must consider the processes that lead to the creation and decay of Hopium similar to our analysis in the salt tank problem. We are given that Utopium decays into Hopium, so the increase in Hopium will be related to the decay of Utopium. Hopium is also radioactive and will naturally decay on its own as well. These processes together determine the rate of change for the amount of Hopium in our system, which is the derivative of \( H(t) \) with respect to the time \( t \). So,

\[
\frac{dH}{dt} = -k_H H - \frac{dU}{dt},
\]

where \( k_H \) is some positive constant related to the half-life of Utopium. Specifically, \( k_H = \frac{\ln(2)}{t_{1/2,H}} = \frac{\ln(2)}{14} \) where \( t_{1/2,H} \) is the half-life of Hopium. The \(-k_H H \) term in the above differential equation represents the radioactive decay of Hopium as usual. The \(-\frac{dU}{dt}\) represents the fact that Utopium decays into Hopium, so any decrease in Utopium implies an equal increase in the amount of Hopium. The minus sign makes sure that the term represents an increase in Hopium. We already have our equation for \( dU/dt \), so we may substitute that into the above differential equation:

\[
\frac{dH}{dt} = -k_H H + k_U U.
\]

Since we already found our equation for \( U(t) \), we may enter that into the differential equation too:

\[
\frac{dH}{dt} = -k_H H + k_U e^{-k_U t}.
\]

The above differential equation is a first order, linear differential equation that we know how to solve using the integrating factor method. First, write the equation in standard form \((H' + a(t)H = b(t))\):

\[
\frac{dH}{dt} + k_H H = k_U e^{-k_U t}.
\]

Next, determine the integrating factor. The integrating factor is

\[
e^\int k_H dt = e^{k_H t}.
\]

Multiply the differential equation in standard form by the integrating factor to get

\[
e^{k_H t} \frac{dH}{dt} + k_H e^{k_H t} H = k_U e^{(k_H - k_U)t}.
\]

Using the product rule of differentiation, we may rewrite the above differential equation as

\[
\frac{d}{dt} \left( e^{k_H t} H \right) = k_U e^{(k_H - k_U)t}.
\]
Integrate. You may use the standard $u$ substitution to calculate the integral for the right hand side.

$$e^{k_H t}H = \frac{k_U}{k_H - k_U}e^{(k_H - k_U)t} + C_3.$$ 

Divide both sides by $e^{k_H t}$. Our general solution for Hopium is

$$H(t) = \frac{k_U}{k_H - k_U}e^{-k_U t} + C_3e^{-k_H t}.$$ 

The value of $C_3$ may be determined with an initial condition. Our implied initial condition for this problem is that we started with zero Hopium, or $H(0) = 0$. So,

$$0 = \frac{k_U}{k_H - k_U} + C_3 \Rightarrow C_3 = -\frac{k_U}{k_H - k_U}.$$ 

The amount of Hopium at time $t$ is given by

$$H(t) = \frac{k_U}{k_H - k_U} [e^{-k_U t} - e^{-k_H t}] = 2 \left[ e^{-k_H t} - e^{-k_U t} \right],$$

where I simplified the fraction of the constant $k$’s knowing that $k_H = \ln(2)/14$ and $k_U = \ln(2)/7$. If you wish, the final solution may also be written as

$$H(t) = 2 \left[ \left( \frac{1}{2} \right)^{t/14} - \left( \frac{1}{2} \right)^{t/7} \right].$$