This assignment is a continuation of the problem from the previous homework assignment:

Given parameters $c > 0$ and $\beta > 0$, show that the Sturm-Liouville boundary value problem

$$
y'' + \lambda y = 0, \quad y = y(x), \quad 0 \leq x \leq c,
$$
y'(0) = \beta y(0),
y'(c) = \beta y(c),
$$

has exactly one negative eigenvalue $\lambda_0$ and that this eigenvalue is independent on $c$. Find $\lambda_0$ and an associated eigenfunction $y_0$. Determine whether $\lambda = 0$ is an eigenvalue. If yes, find an associated eigenfunction.

**Problem 1.** Solve the temperature problem:

$$
u_t = ku_{xx}, \quad u = u(x,t), \quad 0 \leq x \leq \pi, \quad t > 0 \quad (k > 0)
$$
u_x(0,t) = \beta u(0,t),
u_x(\pi,t) = \beta u(\pi,t),
u(x,0) = f(x),
$$

where $f(x)$ is a given continuous function on $[0, \pi]$ and $\beta$ is a positive parameter. Write your answer in the form of an infinite series

$$
u(x,t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t). \quad (1)
$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients $c_n$ in terms of $f$.

**Problem 2.** Assume the initial temperatures are constant: let $f(x) = 1$, for definiteness. Determine the first three terms in the representation (1), that is, eventually, evaluate $c_0, c_1,$ and $c_2$.

**Problem 3.** Given parameters $A, B, C$ (real) and $\beta > 0$, consider the temperature problem with non-homogeneous boundary conditions:

$$
u_t = ku_{xx}, \quad u = u(x,t), \quad 0 \leq x \leq \pi, \quad t > 0 \quad (k > 0)
$$
u_x(0,t) = \beta u(0,t) + A,
u_x(\pi,t) = \beta u(\pi,t) + B,
u(x,0) = Cx.
$$

Reduce it to Problem 1 by virtue of a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x,t)$.

**Problem 4.**

a) Describe the solution $u(x,t)$ to the above non-homogeneous problem in the form similar to (1). Do not evaluate inner products and $L^2$-norms (but write formulas for them using definite integrals).

b) Determine the relationship between the parameters $A, B, C$, under which the solution $u(x,t)$ is constant in time (that is, depends on $x$, only).