Problem 1. Find the best approximation $g$ in the mean on the interval $0 \leq x \leq \pi$ for the function $f(x) = 1$ using linear combinations of $f_1(x) = \sin x$, $f_2(x) = \sin 2x$, $f_3(x) = \sin 3x$.

Then evaluate the error of approximation, that is, $\|f - g\|$ in $L^2[0, \pi]$.

Problem 2. On the interval $[0, \pi]$ find
a) the Fourier sine series for the function $f$ in Problem No. 2(a) on page 12;
b) the Fourier cosine series for the function $f$ in Problem No. 4 on page 13;
c) the Fourier cosine series for the function $f$ in Problem No. 3(a) on page 12.

In addition, in c) write down Parseval’s equality corresponding to this Fourier series and use it to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Problem 3. Let $S_nf(x)$ denote the $n$-th partial sum of the Fourier series in $-\pi \leq x \leq \pi$ for the function defined to be $f(x) = x + 1$ for $x > 0$, $f(x) = 2x - 3$ for $x < 0$, and $f(0) = 0$.

a) Evaluate for each $x \in [-\pi, \pi]$ the limit $S(x) = \lim_{n \to \infty} S_nf(x)$.
b) Sketch the graph of $S$ on the whole real line.
c) Find the values $S(10)$, $S(20)$.

Hint: The function $S$ is $2\pi$-periodic, so it is enough to know its values on $(-\pi, \pi)$.

Problem 4. Check that the function $f(x) = -\log x$ belongs to $L^2[0, 1]$ and find its $L^2$-norm. Then consider its cosine Fourier series in the interval $0 \leq x \leq 1$, that is,

$$-\log x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi nx).$$

Evaluate the sum $\sum_{n=1}^{\infty} a_n^2$ without computing the coefficients $a_n$.

Hint: Apply Parseval’s equality for the cosine Fourier series in $[0, c]$. You will also need to compute $a_0$. 