Problem 1 (7 points). a) Given parameters $c > 0$ and $\beta > 0$, show that the Sturm-Liouville boundary value problem

$$y'' + \lambda y = 0, \quad y = y(x), \quad 0 \leq x \leq c,$$

$$y'(0) = \beta y(0), \quad y'(c) = \beta y(c),$$

has exactly one negative eigenvalue $\lambda_0$ and that this eigenvalue is independent on $c > 0$. Find $\lambda_0$ and an associated eigenfunction $y_0(x)$.

b) Determine whether or not $\lambda = 0$ is an eigenvalue. If yes, find an associated eigenfunction.

Problem 2 (6 points). Solve the temperature problem:

$$u_t = ku_{xx}, \quad u = u(x,t), \quad 0 \leq x \leq \pi, \quad t > 0 \quad (k > 0)$$

$$u_x(0,t) = \beta u(0,t), \quad u_x(\pi,t) = \beta u(\pi,t),$$

$$u(x,0) = f(x),$$

where $f(x)$ is a given continuous function on $[0, \pi]$ and $\beta$ is a positive parameter. Write your answer in the form of an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} c_n y_n(x) T_n(t). \quad (1)$$

Describe the functions $y_n(x)$ and $T_n(t)$ that are involved and indicate how to compute the coefficients $c_n$ in terms of $f$.

Problem 3 (6 points). Assume the initial temperatures are constant: let $f(x) = 1$, for definiteness. Determine the first three terms in the representation (1), that is, eventually, evaluate $c_0$, $c_1$, and $c_2$.

Problem 4 (6 points). Given parameters $A, B, C$ (real) and $\beta > 0$, consider the temperature problem with non-homogeneous boundary conditions:

$$u_t = ku_{xx}, \quad u = u(x,t), \quad 0 \leq x \leq \pi, \quad t > 0 \quad (k > 0)$$

$$u_x(0,t) = \beta u(0,t) + A, \quad u_x(\pi,t) = \beta u(\pi,t) + B,$$

$$u(x,0) = Cx.$$

Reduce it to Problem 2 by virtue of a suitable substitution $u(x,t) = U(x,t) + \Phi(x)$. Indicate new initial temperatures $F(x)$ in the homogeneous problem about $U(x,t)$. 