Problem 1 (7 points). Find all eigenvalues and associated eigenfunctions for the Sturm-Liouville problem
\[ y''(x) + \lambda y(x) = 0, \quad 0 \leq x \leq \pi, \]
\[ y(0) + y'(0) = 0, \]
\[ y(\pi) + y'(\pi) = 0. \]

Problem 2 (6 points). a) Solve the temperature problem in the slab
\[ u_t = k u_{xx}, \quad u = u(x,t), \quad 0 \leq x \leq \pi, \quad t \geq 0 \quad (k > 0) \]
\[ u_x(0, t) = 0, \]
\[ u_x(\pi, t) = 0, \]
\[ u(x, 0) = (1 + \cos^2 x)^2. \]

b) Find the temperatures in the long run: Show that there is a finite limit \( C = \lim_{t \to \infty} u(x, t) \) which is independent of \( x \). Find the constant \( C \).

Note. The initial temperatures may be represented as a cosine polynomial of degree 4.

Problem 3 (6 points). Use the Fourier transform to solve the temperature problem in the upper half-plane
\[ u_t = k u_{xx}, \quad u = u(x,t), \quad -\infty < x < \infty, \quad t \geq 0, \quad u \text{ is bounded}, \]
\[ u(x, 0) = e^{-2x^2}. \]

Hint. First formulate a general theorem about the boundary value problems
\[ u_t = k u_{xx}, \quad u = u(x,t), \quad -\infty < x < \infty, \quad t \geq 0, \quad u \text{ is bounded}, \]
\[ u(x, 0) = f(x). \]

Problem 4 (6 points) Use the Fourier transform to solve the boundary value problem, involving the Laplace equation:
\[ \Delta u = 0, \quad u = u(x, y), \quad x, y \geq 0, \quad u \text{ is bounded}, \]
\[ u_x(0, y) = 0, \]
\[ u(x, 0) = \frac{1}{1+x^2}. \]

Hint. First formulate a general theorem about boundary value problems of the form
\[ \Delta u = 0, \quad u = u(x, y), \quad x, y \geq 0, \quad u \text{ is bounded}, \]
\[ u_x(0, y) = 0, \]
\[ u(x, 0) = f(x). \]

Note that the Fourier transform for the functions \( f(x) = e^{-x^2/(2\sigma^2)} \) and \( f(x) = \frac{1}{1+x^2} \) are known and have been evaluated in class.