Remarks on the Growth of $L^p$-norms of Polynomials

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Abstract. We study the behaviour of constants in Khinchine-Kahane-type inequalities for polynomials in random vectors which have logarithmically concave distributions.

It is well known that polynomials on $\mathbb{R}^n$ of bounded degree satisfy dimension free Khinchine-Kahane-type inequalities with respect to the uniform distribution on convex sets. The best known result, which is due to J. Bourgain [Bou] and which gave an affirmative answer to a conjecture of V. D. Milman, indicates the following: there exist universal constants $t_0 > 0$ and $c \in (0, 1)$ such that, for every convex set $K \subset \mathbb{R}^n$ of volume one, every polynomial $f = f(x_1, \ldots, x_n)$ of degree $d \geq 1$, we have a distribution inequality

$$\mu_K \{ |f| \geq \|f\|_1 t \} \leq \exp \{-tc/d\}, \quad t \geq t_0,$$

where $\mu_K$ is the Lebesgue measure on $K$, and $\|f\|_1$ is $L^1$-norm of $f$ with respect to $\mu_K$. As usual, for $p > 0$, one denotes $\|f\|_p = (\int |f|^p \, d\mu)^{1/p}$ which also refers to some probability measure $\mu$ on a space where the function $f$ is defined. The inequality (1) may also be written in terms of a suitable Orlicz norm. For $\alpha \geq 1$, set $\psi_\alpha(t) = \exp \{t^\alpha\} - 1, \quad t \geq 0$, and introduce the associated norm

$$\|f\|_{\psi_\alpha} = \inf \bigg\{ \lambda > 0 : \int \psi_\alpha(|f|/\lambda) \, d\mu \leq 1 \bigg\}.$$

Then, (1) is equivalent to the inequality

$$\|f\|_{\psi_{c/d}} \leq C^d \|f\|_1,$$

with some universal $c \in (0, 1)$ and $C > 0$. In particular, this yields the equivalence between $L^p$ and $L^1$-norms in the form of the Khinchine-Kahane-type inequality

$$\|f\|_p \leq C(d, p) \|f\|_1,$$

where $C(d, p)$ depends on $d$ and $p$, only. The case $d = 1$, with the best constant $c = 1$ in (2), was settled before by M. Gromov and V. D. Milman [G-M1] (cf. also [M-S]). It is therefore interesting to know whether or not the inequality

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(2) with \(c = 1\) holds true in the general case \(d \geq 1\). The latter is equivalent to the statement that the inequality (3) holds in the range \(p \geq 1\) with constants \(C(d, p) = (Cdp)^d\), for some universal \(C\). In this note, we would like to show, following R. Kannan, L. Lovász and M. Simonovits [K-L-S], a short proof and a refinement of such a statement involving more general classes of probability measures on \(\mathbb{R}^n\).

**Theorem 1.** With respect to an arbitrary log-concave probability measure \(\mu\) on \(\mathbb{R}^n\), for every polynomial \(f\) on \(\mathbb{R}^n\) of degree \(d \geq 1\), we have, for some universal \(C\),

\[
\|f\|_{\psi_1/s} \leq C^d \|f\|_0.
\]  

Here, \(\|f\|_0 = \lim_{p \to 0} \|f\|_p = \exp \left\{ \int \log |f| \, d\mu \right\}\). The log-concavity of \(\mu\) means (cf. [Bor]) that \(\mu\) is supported by some affine subspace \(E \subseteq \mathbb{R}^n\) where it has a logarithmically concave density with respect to Lebesgue measure on \(E\), i.e., a density \(u : E \to [0, +\infty)\) such that, for all \(x, y \in E\), and all \(t, s > 0\) with \(t + s = 1\),

\[
u(tx + sy) \geq u(x)^t u(y)^s.
\]

As a main statement (Theorem 1.3 in [Bou]), J. Bourgain formulated the inequality (2) with, however, a constant \(C\) instead of \(C^d\). As we believe, the power \(d\) was lost when deriving (2) from (1). That the power of a constant cannot be omitted can be seen on the example of the set \(K = [0, 1]\) with the function \(f = 1\) and with growing \(d\). On the other hand, the power can be hidden, if one wishes to rewrite for example the inequality (4) in the equivalent form as

\[
\| |f|^{1/d} \|_{\psi_1} \leq C \| |f|^{1/d} \|_0
\]

(with the same constant). This inequality is very much similar to what is known for norms on \(\mathbb{R}^n\) in the place of \(|f|^{1/d}\) (cf. [L]).

To prove (4), we use Theorem 2.7 from [K-L-S] which is based on the localization lemma of L. Lovász and M. Simonovits [L-S]. It reduces multi-dimensional Khinchine-Kahane-type inequalities with respect to log-concave measures to dimension one that was illustrated in [K-L-S] on the example of inequalities of the form (3) in the linear case \(d = 1\). It was also mentioned there that the method applies as well to the general case \(d \geq 1\) and allows one to recover Bourgain's theorem. This is completely true up to a remark that the one-dimensional case in (3) still requires and deserves a special careful consideration, since it determines behaviour of constants as a function of \(d\) and \(p\). The localization lemma of [L-S] has also applications to other types of inequalities. In connection with isoperimetric inequalities on the sphere, a kind of localization technique was developed by M. Gromov and V. D. Milman in [G-M2], cf. also [A].

For the Gaussian measure on the real line \(\mathbb{R}\), using expansions over Hermit polynomials, the inequality (3) with \(p = 2\) was studied by Yu. V. Prokhorov.
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[P1]; he obtained as well similar inequalities when $\mu$ belongs to the family of $\Gamma(\alpha)$-distributions with the parameter $\alpha$ growing with the degree $d$, cf. [P2]. As shown in these works, in both cases, the constants $C(d, 2)$ grow exponentially, and this fact cannot be recovered by an inequality such as (2) or (4). Combining Prokhorov’s approach with the localization method, one can prove:

**Theorem 2.** With respect to an arbitrary log-concave probability measure $\mu$ on $\mathbb{R}^n$, for every polynomial $f$ on $\mathbb{R}^n$ of degree $d \geq 1$, the Khinchine-Kahane-type inequality (3) holds with

$$C(d, p) = p^{C_d}, \quad p \geq 2,$$

where $C$ is universal.

Consider the example of the exponential measure $\mu = \nu$ on $\mathbb{R}$ with density $d\nu(x) = e^{-x}, \ x > 0$, and take $f(x) = x^d$. Then, $\|f\|_p = \Gamma(dp + 1)^{1/p}$ and, by Stirling’s formula,

$$c_1 \frac{p^d}{d(d-1)/2p} \|f\|_1 \leq \|f\|_p \leq c_2 \frac{p^d}{d(d-1)/2p} \|f\|_1,$$

for some $c_1, c_2 > 0$. Therefore, when $p$ is fixed and $d$ grows, the constant $C(d, p) = p^{C_d}$ gives a correct exponential rate of increase.

To compare with Theorem 1, note that, up to a universal constant, the inequality (4) is equivalent to

$$\|f\|_p \leq (Cd)^d \|f\|_0, \quad p \geq 1/d. \quad (5)$$

Here, since we have replaced $\|f\|_1$ with a smaller quantity $\|f\|_0$, the constants have a different order. Again, for $f(x) = x^d$, we have $\|f\|_0 = \|x\|_0^d$ with respect to $\nu$, so

$$(c_1 dp)^d \|f\|_0 \leq \|f\|_p \leq (c_2 dp)^d \|f\|_0,$$

for some $c_1, c_2 > 0$. One could also test sharpness of (5) on the example of the uniform distribution on the $\ell^1$-ball in $\mathbb{R}^n$ for $f(x) = x_1^d$ and with growing dimension $n$.

**Proof of Theorem 1.** First let us comment on the one-dimensional case in (5). Motivated by the results of [P1][P2], inequalities of the form (5), with respect to an arbitrary probability measure $\mu$ on $\mathbb{R}$, were studied in [B-G]. As was observed there, given $p > 0$ and $d \geq 1$, the optimal constant $C = C(d, p; \mu)$ in

$$\|f\|_p \leq C \|f\|_0, \quad (6)$$

where $f$ is an arbitrary polynomial of degree $d$ on $\mathbb{R}$ with complex coefficients, is given by

$$C^{1/d} = \sup_{z \in \mathbb{C}} \frac{\|x - z\|_d}{\|x - z\|_0}. \quad (7)$$
Since the argument is straightforward, let us recall it. A remarkable feature of the functional $\|f\|_{0}$ is its multiplicativity property: $\|f_{1}\cdots f_{d}\|_{0} = \|f_{1}\|_{0}\cdots\|f_{d}\|_{0}$. Therefore, writing $f(z) = A(z - z_{1})\cdots(z - z_{d})$ and applying Hölder's inequality, we get

$$\|f\|_{p} \leq A \prod_{k=1}^{d} \|x - z_{k}\|_{pd} \leq CA \prod_{k=1}^{d} \|x - z_{k}\|_{0} = C\|f\|_{0},$$

where $C$ is defined according to (7). This proves (6) with this constant which cannot be improved as the example of the polynomials $f(z) = z - z$ shows.

It might be helpful to note that the sup in (7) can be restricted to the real line $\mathbb{R}$. Indeed, write $z = a + bi$ so that $\|x - z\|^{2} = \xi + \tau$ with $\xi = (z - a)^{2} \geq 0$ and $\tau = b^{2} \geq 0$. As can easily be verified by differentiation, for any $p > q$, the function of the form $g(t) = \|\xi + \tau\|_{p}/\|\xi + \tau\|_{q}$ is non-increasing in $t \geq 0$, hence it is maximized at $t = 0$. Therefore, as a function of $b$, the value of

$$\frac{\|x - z\|_{p}}{\|x - z\|_{q}} = \frac{\|\xi + \tau\|_{p/2}^{1/2}}{\|\xi + \tau\|_{q/2}^{1/2}}$$

is maximized at $b = 0$. So, we may apply this observation with $q = 0$ to (7).

In the particular case, where $\mu$ is log-concave, the right hand side of (7) can be bounded by a quantity which is independent of $\mu$ and grows like $Cd\mu$. Indeed, after shifting, one needs to estimate an optimal constant $C$ in

$$\|\|z\|\|_{dp} \leq C^{1/d} \|z\|_{0}, \quad p \geq 1/d,$$

with respect to a log-concave measure on $\mathbb{R}$. The fact that such an inequality holds for an arbitrary log-concave measure $\mu$ on $\mathbb{R}^{n}$, and for an arbitrary norm $\|z\|$ instead of $|z|$ was established by R. Latała [L] (cf. also [B] and [Gu] for different proofs). More precisely, he showed that, for some universal $C_{1}$, we always have

$$\|\|z\|\|_{1} \leq C_{1} \|\|z\|\|_{0}.$$

On the other hand, it had been known, as an application of Borell's lemma [Bor], that, for $p \geq 1$,

$$\|\|z\|\|_{p} \leq C_{2}p \|\|z\|\|_{1}.$$

These two inequalities give $\|\|z\|\|_{p} \leq C_{0}p \|\|z\|\|_{0}$. Thus, the constant in (8), $C^{1/d}$, is bounded by $C_{0}dp$, for some universal $C_{0}$. This proves (5) with $C = C_{0}$.

To treat the multidimensional case in (5), one may assume that $\mu$ is absolutely continuous. As proved in [K-L-S], Theorem 2.7, given non-negative continuous functions $f_{1}, f_{2}, f_{3}$ and $f_{4}$ on $\mathbb{R}^{n}$, and the numbers $\alpha, \beta > 0$, the following two properties are equivalent:
(a) for every absolutely continuous log-concave probability measure \( \mu \) on \( \mathbb{R}^n \),

\[
\left( \int f_1 \, d\mu \right)^{\alpha} \left( \int f_2 \, d\mu \right)^{\beta} \leq \left( \int f_3 \, d\mu \right)^{\alpha} \left( \int f_4 \, d\mu \right)^{\beta};
\]

(9)

(b) for every non-degenerate interval \( \Delta \subset \mathbb{R}^n \) with directional vector \( v \in \mathbb{R}^n \), and for every \( \lambda \in \mathbb{R} \),

\[
\left( \int_{\Delta} e^{\lambda(v,z)} f_1(x) \, dx \right)^{\alpha} \left( \int_{\Delta} e^{\lambda(v,z)} f_2(x) \, dx \right)^{\beta} \leq \left( \int_{\Delta} e^{\lambda(v,z)} f_3(x) \, dx \right)^{\alpha} \left( \int_{\Delta} e^{\lambda(v,z)} f_4(x) \, dx \right)^{\beta},
\]

where \( dx \) stands for the Lebesgue measure on \( \mathbb{R}^n \).

Thus, (9) reduces to the case where the measure \( \mu \) is supported by a line and, moreover, represents there the restriction of the exponential measure \( e^{\lambda(v,z)} \, dx \) to the interval \( \Delta \). Applying the above equivalence to \( \alpha = 1/p \), \( \beta = 1/q \) and to the functions \( f_1 = |f|^p \), \( f_4 = C |f|^q \) \((C > 0)\), and \( f_2 = f_3 = 1\), R. Kannan, L. Lovász and M. Simonovits made the following striking conclusion which we state here as a lemma.

**Lemma 1.** Let \( p > q > 0 \) and \( C \geq 1 \). Given a continuous function \( f \) on \( \mathbb{R}^n \), the inequality

\[
\|f\|_p \leq C \|f\|_q
\]

holds true with respect to all log-concave probability measures \( \mu \) on \( \mathbb{R}^n \) if and only if it holds on all intervals in \( \mathbb{R}^n \) with respect to the normalized exponential measures.

By continuity, one can clearly consider in (10) the case \( q = 0 \), as well.

Now, if \( f \) is polynomial, its restriction to every line is again a polynomial of the same degree but of one variable. Since the restrictions of the exponential measures are log-concave, the inequality (10) thus reduces to the one-dimensional case.

One may therefore conclude that the inequality (5) holds with \( C = C_0 \) for all polynomials \( f \) on \( \mathbb{R}^n \) of degree \( d \) in the range \( p \geq 1/d \). Applying (5) to \( p = k/d \), \( k = 1, 2, \ldots \), we get

\[
\|f\|_{1/d} \leq C_0 k \|f\|_0^{1/d}.
\]

Finally, by Taylor's expansion and using \( k^k \leq e^k k! \), we obtain that

\[
\int \psi_{1/d} \left( \frac{f}{(2eC_0)^d \|f\|_0} \right) \, d\mu \leq 1.
\]

**Proof of Theorem 2.** First, we consider the growth of the constants \( C(d, 2) \) in (3) in the case \( p = 2 \). We will now use Lemma 1, with \( p = 2 \) and \( q = 1 \),
in full volume that gives more than just a reduction of the multidimensional inequality (10) to dimension one. Let $\Delta$ be a non-degenerate interval in $\mathbb{R}^n$ with endpoints $a, b$ and with the directional vector $v = (b - a)/|b - a|$. Then, with respect to the normalized exponential measures on $\Delta$, the norms in (10) are given by

$$
\|f\|_p^p = \frac{1}{\int_0^{b-a} e^{\lambda x} \, dx} \int_0^{b-a} |f(a + xv)|^p e^{\lambda x} \, dx.
$$

When $f$ is a polynomial on $\mathbb{R}^n$, $f(a + xv)$ represents a polynomial in $x \in \mathbb{R}$ of the same degree. Moreover, after rescaling, it suffices to consider the case $\lambda = -1$. Therefore, $C(d, 2)$ is the optimal constant $C$ in the inequality

$$
\left( \int |f|^2 \, d\nu_u \right)^{1/2} \leq C \int |f| \, d\nu_u
$$

(11)

for the class of all polynomials $f$ on $\mathbb{R}$ of degree $d$ with respect to all measures $\nu_u$ on $(0, +\infty)$ with densities

$$
d\nu_u(x) = \frac{e^{-x}}{1 - e^{-u}} 1_{(0,u)}(x), \quad u > 0.
$$

The limit case represents the exponential measure $\nu_{\infty} = \nu$ on $(0, +\infty)$ with density $e^{-x}$, $x > 0$. For a related family of densities, $x^\alpha e^{-x}/\Gamma(\alpha + 1)$, the inequality (11), with exponentially increasing constants, was proved by Yu. V. Prokhorov [P2]. He assumed that $\alpha \geq c_0d$ (for a numerical $c_0$), but his approach proposed before in [P1] actually works in a more general situation and can be applied in particular to the measures $\nu_u$. Below, to prove (11), we follow Prokhorov’s scheme of the proof and simplify his argument about Laguerre’s polynomials.

**Step 1:** $0 < u < 8d$.

Let $f$ be an arbitrary polynomial on $\mathbb{R}$ of degree $d$ (with real coefficients) such that $\|f\|_2 = 1$ where $L^2$-norm is understood with respect to $\nu_u$. Let $x_0 \in [0, u]$ be such that $|f(x_0)| = \|f\|_\infty = \max_{x \in [0, u]} |f(x)|$, and assume, without loss of generality, that $f(x_0) > 0$. By Taylor’s expansion and by Markov’s inequality $\|f^\prime\|_\infty \leq \frac{2d^2}{u} \|f\|_\infty$, we get, for every point $x \in [0, u]$,

$$
f(x) \geq f(x_0) - \|f^\prime\|_\infty |x - x_0|
$$

$$
\geq f(x_0) - \frac{2d^2}{u} \|f\|_\infty |x - x_0| = \left(1 - \frac{2d^2}{u} |x - x_0|\right) \|f\|_\infty.
$$

Therefore, in the interval $\delta = [x_1, x_2] \equiv [x_0 - u/(4d^2), x_0 + u/(4d^2)] \cap [0, u]$, we have $f(x) \geq \frac{1}{2} \|f\|_\infty$ so that

$$
\|f\|_1 \geq \int_\delta f(x) \, d\nu_u(x) \geq \frac{1}{2} \|f\|_\infty \nu_u(\delta).
$$
In addition, since \( x_2 - x_1 \geq u/(4d^2) \), for some middle point \( z_3 \in [x_1, x_2] \), we get

\[
\nu_u(\delta) = \frac{e^{-z_1} - e^{-z_2}}{1 - e^{-u}} = \frac{x_2 - x_1}{1 - e^{-u}} e^{-z_1} = \frac{1}{4d^2} \frac{u}{1 - e^{-u}} e^{-8d} \geq \frac{1}{4d^2} e^{-8d}.
\]

Hence, \( 8d^2 e^{8d} ||f||_1 \geq ||f||_{\infty} \geq ||f||_2 \) so that (11) is fulfilled with \( C = 8d^2 e^{8d} \).

The second step requires some preparation.

**Lemma 2.** For every polynomial \( f \) on \( \mathbb{R} \) of degree \( d \geq 1 \),

\[
\int_{8d}^{+\infty} |f(x)|^2 e^{-x} dx \leq \frac{1}{2} \int_0^{+\infty} |f(x)|^2 e^{-x} dx.
\]

**Proof.** Assume that \( \int_0^{+\infty} |f(x)|^2 e^{-x} dx = ||f||_2^2 = 1 \) (with respect to \( \nu \)) and introduce the Laguerre polynomials

\[
L_k(z) = \frac{e^z}{k!} \frac{d^k}{dx^k} (x^k e^{-x}) = \sum_{j=0}^k (-1)^j C_k^j z^j, \quad k = 0, 1, \ldots \tag{12}
\]

They form a complete orthonormal system of functions in \( L^2(\nu) \) so that there exists a representation \( f = \sum_{k=0}^{d} a_k L_k \) with \( \sum_{k=0}^{d} |a_k|^2 = 1 \). Hence, \( ||f||_2^2 \leq \sum_{k=0}^{d} ||L_k||_2^2 \) so that

\[
||f||_2^4 = \sum_{k=0}^{d} ||L_k||_2^2 \leq \sum_{k=0}^{d} ||L_k||_4^2 = \sum_{k=0}^{d} ||L_k||_2^4.
\]

According to (12) and since \((4j)! \leq 4^{4j} j!^4 \), we get

\[
||L_k||_4 \leq \sum_{j=0}^{k} C_k^j \frac{||x^j||_4}{j!} = \sum_{j=0}^{k} C_k^j \frac{(4j)!^{1/4}}{j!^4} \leq \sum_{j=0}^{k} C_k^j 4^{j/4} = 5^k.
\]

Thus, by (13), \( ||f||_2^4 \leq \sum_{k=0}^{d} 25^k < \frac{25^{d+1}}{24} \). Now, by Cauchy-Schwarz inequality,

\[
\int_{8d}^{+\infty} |f(x)|^2 e^{-x} dx \leq ||f||_2^2 \int_{[8d, +\infty)} ||L_k||_2^2 dx \leq \frac{25^{d+1}}{24} e^{-4d} < \frac{1}{2}.
\]

**Step 2:** \( u \geq 8d \).

Again, let \( f \) be an arbitrary polynomial on \( \mathbb{R} \) of degree \( d \) such that \( ||f||_2 = 1 \) where \( L^2 \)-norm is with respect to \( \nu_u \). Now, our basic interval will be \([0, 8d]\). Let \( z_0 \) be a point of maximum of \( |f| \) on this interval, and assume that \( f(z_0) > 0 \). Again, by Taylor’s expansion and by Markov’s inequality, for every point \( x \in [0, 8d] \),

\[
f(x) \geq f(z_0) - ||f'||_\infty |x - z_0| \geq \left( 1 - \frac{d}{4} |x - z_0| \right) ||f||_\infty
\]
\( L^\infty \)-norm is taken on \([0, 8d]\). Therefore, in the interval
\[
\delta = [x_1, x_2] \equiv [x_0 - 2/d, x_0 + 2/d] \cap [0, 8d]
\]
we have \( f(x) \geq \frac{1}{2} \|f\|\infty \) so that
\[
\|f\|_1 \geq \int_\delta f(x) \, d\nu_u(x) \geq \frac{1}{2} \|f\|\infty \nu_u(\delta). \quad (14)
\]
On the other hand, since \( u \geq 8d \) and by Lemma 2,
\[
\int_0^{8d} |f|^2 \, d\nu_u = \frac{1}{1 - e^{-u}} \int_0^{8d} |f(x)|^2 e^{-x} \, dx \geq \frac{1}{2(1 - e^{-u})} \int_0^{+\infty} |f(x)|^2 e^{-x} \, dx
\]
\[
\geq \frac{1}{2(1 - e^{-u})} \int_0^{u} |f(x)|^2 e^{-x} \, dx = \frac{1}{2}.
\]
Therefore, \( \frac{1}{2} \leq \|f\|_\infty^2 \nu_u(\{0, 8d\}) \leq \|f\|_\infty^2 \). Combining with (14), we get \( \|f\|_1 \geq \frac{1}{2\sqrt{2}} \nu_u(\delta) \). Using \( x_2 - x_1 \geq 2/d \), we obtain that, for some middle point \( x_3 \in [x_1, x_2] \),
\[
\nu_u(\delta) = \frac{e^{x_3} - e^{x_1}}{1 - e^{-u}} = \frac{x_2 - x_1}{1 - e^{-u}} e^{-x_3} \geq \frac{2}{d} e^{-8d}.
\]
Hence, \( \sqrt{2}d e^{8d} \|f\|_1 \geq \|f\|_2 \) so that (11) is fulfilled with \( C = \sqrt{2}d e^{8d} \). Note that this constant is majorized by the constant \( 8d^2 e^{8d} (\leq e^{11d}) \) obtained on the first step.

Thus, for every polynomial \( f \) on \( \mathbb{R}^n \) of degree \( d \), with respect to an arbitrary log-concave probability measure \( \mu \) on \( \mathbb{R}^n \),
\[
\|f\|_2 \leq e^{11d} \|f\|_1. \quad (15)
\]
It remains to consider the general case \( p \geq 2 \), in order to complete the proof of Theorem 2. One can iterate an inequality of the form (15), \( \|f\|_p^2 \leq A^d \|f\|_2^2 \), starting from \( f \) and successively applying it to the polynomials \( f, f^2, f^4, \ldots, f^{2^k} \). This yields
\[
\|f\|_2^{2^k} \leq A^{k2^{k-1}d} \|f\|_1^{2^k}, \quad k \geq 1.
\]
Assume \( \|f\|_1 = 1 \) and pick up \( k \geq 1 \) such that \( 2^k \leq p < 2^{k+1} \). Then,
\[
\|f\|_p \leq \|f\|_2^{2^{k+1}} \leq A^{(k+1)d/2} \leq A^{kd} \leq p^{d\log A/\log 2}.
\]
According to (15), we can apply these estimates with \( A = e^{22} \) and thus get the statement of Theorem 2 with \( C = 22/\log 2 \).

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