Chapter 11
Asymptotic Behavior of Rényi Entropy in the Central Limit Theorem

Sergey G. Bobkov and Arnaud Marsiglietti

Abstract We explore an asymptotic behavior of Rényi entropy along convolutions in the central limit theorem with respect to the increasing number of i.i.d. summands. In particular, the problem of monotonicity is addressed under suitable moment hypotheses.

Keywords Rényi entropy · Central limit theorem

2010 Mathematics Subject Classification Primary 60E, 60F

11.1 Introduction

Given a (continuous) random variable X with density p, the associated Rényi entropy and Rényi entropy power of index r ($1 < r < \infty$) are defined by

$$h_r(X) = -\frac{1}{r-1}\log\int_{-\infty}^{\infty} p(x)^r\,dx, \qquad N_r(X) = e^{2h_r(X)} = \left(\int_{-\infty}^{\infty} p(x)^r\,dx\right)^{-\frac{2}{r-1}}.$$

Being translation invariant and homogeneous of order 2, the functional N_r is similar to the variance and is often interpreted as measure of uncertainty hidden in the distribution of X. Another representation

$$N_r(X)^{-\frac{1}{2}} = \left(\mathbb{E}\,p(X)^{r-1}\right)^{\frac{1}{r-1}}$$

S. G. Bobkov
School of Mathematics, University of Minnesota, Minneapolis, MN, USA
e-mail: bobko001@umn.edu

A. Marsiglietti (✉)
Department of Mathematics, University of Florida, Gainesville, FL, USA
e-mail: a.marsiglietti@ufl.edu

N. Gozlan et al. (eds.), *High Dimensional Probability VIII*,
Progress in Probability 74, https://doi.org/10.1007/978-3-030-26391-1_11

shows that N_r is non-increasing in r, so that $0 \le N_\infty \le N_r \le N_1 \le \infty$. Here, for the extreme indexes, the Rényi entropy power is defined by the monotonicity,

$$N_\infty(X) = \lim_{r\uparrow\infty} N_r(X) = \|p\|_\infty^{-2}, \qquad N_1(X) = \lim_{r\downarrow 1} N_r(X) = e^{2h_1(X)},$$

where $\|p\|_\infty$ is the essential supremum of $p(x)$. In the case $r = 1$, we arrive at the Shannon differential entropy $h_1(X) = h(X) = -\int p(x) \log p(x)\, dx$ with entropy power $N_1 = N = e^{2h}$ (provided that $N_r(X) > 0$ for some $r > 1$).

Much of the analysis about the Shannon and Rényi entropies is focused on the behavior of these functionals on convolutions, i.e., for sums $S_n = X_1 + \cdots + X_n$ of independent random variables (including a multidimensional setting). First, let us recall a fundamental entropy power inequality, which may be written in terms of the normalized sums $Z_n = S_n/\sqrt{n}$ as

$$N(Z_n) \ge \frac{1}{n} \sum_{k=1}^{n} N(X_k). \tag{11.1}$$

There are also some extensions of this relation to the Rényi case (cf. [4, 5, 9, 10]).

When X_k's are independent and identically distributed (i.i.d.), with mean zero and variance one, the central limit theorem (CLT) asserts that $Z_n \Rightarrow Z$ with weak convergence in distribution to the Gaussian limit $Z \sim N(0, 1)$. In this case, the right-hand side of (11.1) is constant, while the sequence on the left is monotone, as was shown by Artstein, Ball, Barthe and Naor [1], cf. also [12] (the inequality (11.1) itself ensures that $N(Z_n)$ are non-decreasing along the values $n = 2^l$). Moreover, by another important result due to Barron [2], we have the entropic CLT: $N(Z_n)$ are convergent to the entropy power $N(Z)$, as long as $N(Z_{n_0}) > 0$ for some n_0.

These results give rise to a number of natural questions about an asymptotic behavior of the Rényi entropy powers $N_r(Z_n)$. In particular, when do they converge to $N_r(Z)$, and if so, what is the rate of convergence? Is the monotonicity still true? As we will see, such questions may be studied, at least partially, under suitable moment assumptions.

Let us state a few observations in these directions, assuming throughout that $X, X_1, X_2, \dots$ are i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathrm{Var}(X) = 1$. Put $\beta_s = \mathbb{E}\,|X|^s$ for real $s \ge 2$. In order to describe necessary and sufficient conditions for the convergence of the Rényi entropies in the CLT, we also introduce the common characteristic function

$$f(t) = \mathbb{E}\, e^{itX} \qquad (t \in \mathbb{R}).$$

Theorem 11.1.1 *Given $1 < r \le \infty$, we have the convergence $N_r(Z_n) \to N_r(Z)$ or equivalently $h_r(Z_n) \to h_r(Z)$ as $n \to \infty$, if and only if*

$$\int_{-\infty}^{\infty} |f(t)|^\nu \, dt < \infty \quad \mathit{for\ some}\ \ \nu \ge 1. \tag{11.2}$$

Equivalently, this holds if and only if Z_n have bounded densities for all (some) n large enough.

This characterization coincides with the one for the uniform local limit theorem due to Gnedenko, cf. [11]. Since (11.2) is equivalent to the property that Z_n have bounded and hence bounded C^k-smooth densities for any fixed k and all n large enough, it is often referred to as the smoothing condition. In general, (11.2) is stronger than what is needed in the entropic case $r = 1$. In this connection, let us note that there is still no explicit description such as (11.2) for the validity of the entropic CLT in terms of the characteristic function $f(t)$.

Once (11.2) is fulfilled, one may ask about the rate of convergence in Theorem 11.1.1, which may be guaranteed assuming that the absolute moment β_s is finite for some $s > 2$. Moreover, in this case one may obtain asymptotic expansions for $N_r(Z_n)$ in powers of $1/n$ similarly to the entropic expansions derived in [8]. They involve the moments of X up to order $m = [s]$, or equivalently—the cumulants

$$\gamma_k = i^{-k}\,(\log f)^{(k)}(0), \qquad k = 1, \dots, m.$$

In the Gaussian case $X \sim N(0,1)$, all cumulants are vanishing, starting with $k = 2$. In the general case, they indicate how close a given distribution to the normal. As for the asymptotic behavior of Rényi's entropies, it turns out that a special role is played by the quantity

$$b = b(r) = -\frac{1}{r}\left[\frac{2-r}{12}\,\gamma_3^2 + \frac{r-1}{8}\,\gamma_4\right].$$

Here, $\gamma_3 = \mathbb{E}X^3$ and $\gamma_4 = \mathbb{E}X^4 - 3$, while for the extreme indexes, one may just put

$$b(1) = \lim_{r \to 1} b(r) = -\frac{1}{12}\,\gamma_3^2, \qquad b(\infty) = \lim_{r \to \infty} b(r) = \frac{1}{12}\,\gamma_3^2 - \frac{1}{8}\,\gamma_4.$$

This can be seen from the following assertion.

Theorem 11.1.2 *Suppose that the smoothing condition (11.2) is fulfilled. If β_s is finite for $2 \le s < 4$, then for any $1 < r < \infty$,*

$$h_r(Z_n) = h_r(Z) + o(n^{-\frac{s-2}{2}}), \qquad N_r(Z_n) = N_r(Z) + o(n^{-\frac{s-2}{2}}). \tag{11.3}$$

Moreover, in case $4 \le s < 6$,

$$\begin{aligned} h_r(Z_n) &= h_r(Z) + b\,n^{-1} + o(n^{-\frac{s-2}{2}}), \\ N_r(Z_n) &= N_r(Z)\left(1 + 2b\,n^{-1}\right) + o(n^{-\frac{s-2}{2}}). \end{aligned} \tag{11.4}$$

This assertion remains valid in the entropic case $r = 1$ as well (with a slight logarithmic improvement in the remainder o-term, cf. [8]). In case $s = 6$, the remainder term may be improved to $O(n^{-2})$, and in fact, one may add quadratic terms to get an expansion

$$h_r(Z_n) = h_r(Z) + b\,n^{-1} + b_2 n^{-2} + o(n^{-2}) \tag{11.5}$$

with some functional $b_2 = b_2(r)$ depending also on γ_5 and γ_6. Regardless of its value, one may therefore conclude about an eventual monotonicity of $N_r(Z_n)$ based on the sign of b. Moreover, the above expansions continue to hold for $r = \infty$, so that this case may be included as well.

Theorem 11.1.3 *Suppose that the smoothing condition* (11.2) *is fulfilled, and let β_6 be finite. Given $1 < r \le \infty$, there exists $n_0 \ge 1$ such that the sequence $N_r(Z_n)$ is increasing for $n \ge n_0$, whenever $b(r) < 0$, that is, if*

$$\frac{2-r}{3}\,\gamma_3^2 + \frac{r-1}{2}\,\gamma_4 > 0 \quad (1 < r < \infty), \qquad \gamma_4 > \frac{2}{3}\,\gamma_3^2 \quad (r = \infty).$$

This sequence is decreasing for $n \ge n_0$, if $b(r) > 0$.

In particular, under the last condition $\gamma_4 > \frac{2}{3}\gamma_3^2$, the sequence $N_r(Z_n)$ is eventually increasing for any fixed $r \ge 1$. For example, this holds for $X = \frac{\xi-\alpha}{\sqrt{\alpha}}$, where the random variable ξ has a Gamma distribution with α degrees of freedom (in which case $\gamma_3 = 2/\sqrt{\alpha}$ and $\gamma_4 = 6/\alpha$).

On the other hand, if X is uniformly distributed in the interval $(-\sqrt{3}, \sqrt{3})$, then $\gamma_3 = 0$, $\gamma_4 = -6/5$, so $N_r(Z_n)$ is eventually decreasing for any $r > 1$, although the opposite property takes place for $r = 1$.

The paper is organized as follows. We start with the proof of Theorem 11.1.1 (Sect. 11.2), and then collect together basic results on Edgeworth expansions for densities p_n of Z_n (Sect. 11.3). They are used in Sects. 11.4–11.5 to construct a formal asymptotic expansion for L^r-norms of p_n in powers of $1/n$ up to order $[\frac{m-2}{2}]$ with remainder term as in (11.3)–(11.4). One particular case, where the first moments of X agree with those of $Z \sim N(0,1)$, is discussed separately in Sect. 11.6, while the range $4 \le s \le 8$ in such expansion is treated in Sect. 11.7. The transition to the Rényi entropy is performed in Sect. 11.8, where Theorem 11.1.2 is proved. Some comparison with the entropic CLT is given in Sect. 11.9, with remarks leading to Theorem 11.1.3 for finite r. Finally, the index $r = \infty$ is treated separately in Sect. 11.10. We refer to [6] for an extended version of the article where more computational details are provided.

11.2 Proof of Theorem 11.1.1

From now on, let $X, X_1, X_2, \dots$ be i.i.d. random variables with $\mathbb{E}X = 0$ and $\mathrm{Var}(X) = 1$, for which we define the normalized sums

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}, \qquad n = 1, 2, \dots$$

First, let us recall Gnedenko's uniform local limit theorem. Assuming the smoothing condition (11.2), it asserts that, for all n large enough, the random variables Z_n have bounded densities p_n, and moreover, in that case as $n \to \infty$,

$$\sup_x |p_n(x) - \varphi(x)| \to 0. \tag{11.6}$$

Here, as usual,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \qquad (x \in \mathbb{R})$$

denotes the density of the standard normal random variable Z. Clearly, the property (11.6) is also necessary for the uniform boundedness of p_n's.

Let us explain the equivalence of the two conditions—in terms of the characteristic function as in (11.2), and in terms of densities (via the existence of a bounded density). Since $|f(t)| \le 1$ for all t, the property (11.2) is getting weaker for growing ν, so it is sufficient to consider integer values of ν. Since Z_n has characteristic function

$$f_n(t) = \mathbb{E}\, e^{itZ_n} = f(t/\sqrt{n})^n,$$

(11.2) implies that Z_n has a bounded, continuous density p_n for $n = \nu$, by the Fourier inversion formula. Hence the same is true for all $n \ge \nu$, by the convolution character of the distributions of Z_n. Conversely, suppose that Z_n has a bounded density p_n for $n = n_0$. This implies that $p_n \in L^r(\mathbb{R})$ for any $r \ge 1$, with norm

$$\|p_n\|_r = \left(\int_{-\infty}^{\infty} p_n(x)^r\, dx\right)^{1/r},$$

and in particular $p_n \in L^2(\mathbb{R})$. By Plancherel's theorem, the characteristic function f_n is also in $L^2(\mathbb{R})$. But this means that (11.2) is fulfilled with $\nu = 2n_0$.

Also note that, under the condition (11.2), we have $f_\nu(t) \to 0$ as $t \to \infty$ (the Riemann-Lebesgue lemma), and thus $f(t) \to 0$. Hence, (11.2) represents a sharpening of the Cramér condition $\limsup_{t\to\infty} |f(t)| < 1$, which is used to establish a number of asymptotic results related to the CLT. In particular, using the Fourier inversion formula, one can easily obtain (11.6) and actually a sharper statement such as

$$\sup_x \, (1 + x^2) \, |p_n(x) - \varphi(x)| \to 0 \qquad (n \to \infty). \tag{11.7}$$

Proof of Theorem 11.1.1 First, let $r = \infty$. As explained, the smoothing condition (11.2) implies the uniform local limit theorem (11.6). In turn, the latter yields $\|p_n\|_\infty \to \|\varphi\|_\infty$, that is, $N_\infty(Z_n) \to N_\infty(Z)$ as $n \to \infty$. Conversely, this convergence ensures that $N_\infty(Z_n) > 0$ for all n large enough, that is, $\|p_n\|_\infty < \infty$. As was also emphasized, this implies (11.2).

Now, let $1 < r < \infty$. If $N_r(Z_n) \to N_r(Z)$ as $n \to \infty$, then $N_r(Z_n) > 0$ for all n large enough, say $n \geq n_0$. Equivalently, for such n, Z_n have densities p_n with $\|p_n\|_r < \infty$. If $r \geq 2$, then $\|p_n\|_2 \leq 1 + \|p_n\|_r < \infty$, so that p_n and therefore f_n are in $L^2(\mathbb{R})$. This means that (11.2) is fulfilled for $\nu = 2n_0$. In the case $1 < r < 2$, one may apply the Hausdorff-Young inequality

$$\|\hat{u}\|_{r'} \leq \|u\|_r, \qquad \text{where} \quad \hat{u}(t) = \int_{-\infty}^{\infty} e^{2\pi i t x} \, u(x) \, dx, \quad r' = \frac{r}{r-1}.$$

It implies that $\|f_n\|_{r'} \leq \sqrt{2\pi} \, \|p_n\|_r < \infty$, which means that (11.2) is fulfilled for $\nu = r' n_0$.

Thus, the smoothing condition (11.2) is indeed necessary. To argue in the other direction, we apply the uniform local limit theorem: For all $n \geq n_0$ large enough, Z_n have densities p_n, bounded by a constant M and moreover, the relation (11.6) holds true, i.e.,

$$\sup_x \left| p_n(x)^r - \varphi(x)^r \right| \leq \varepsilon_n \to 0 \qquad (n \to \infty). \tag{11.8}$$

For a given $\varepsilon > 0$, applying the usual central limit theorem, one may pick up $T > 0$ such that

$$\mathbb{P}\{|Z_n| > T\} + \mathbb{P}\{|Z| > T\} < \varepsilon, \qquad n \geq n_1 \geq n_0.$$

Hence

$$\int_{|x|>T} p_n(x)^r \, dx \;\leq\; M^{r-1} \int_{|x|>T} p_n(x) \, dx \;=\; M^{r-1} \, \mathbb{P}\{|Z_n| > T\} \;<\; M^{r-1} \varepsilon,$$

and similarly for $\varphi(x)$. Hence

$$\left| \int_{|x|>T} p_n(x)^r \, dx - \int_{|x|>T} \varphi(x)^r \, dx \right| < M^{r-1} \varepsilon. \tag{11.9}$$

On the other hand, by (11.8),

$$\left| \int_{|x|\le T} p_n(x)^r\,dx - \int_{|x|\le T} \varphi(x)^r\,dx \right| \le \int_{|x|\le T} |p_n(x)^r - \varphi(x)^r|\,dx \le 2T\varepsilon_n \le \varepsilon,$$

where the last inequality holds for all $n \ge n_2$ with some $n_2 \ge n_1$. Together with (11.9), we get

$$\big|\, \|p_n\|_r^r - \|\varphi\|_r^r \big| < (M^{r-1}+1)\,\varepsilon, \qquad n \ge n_2.$$

That is, $\|p_n\|_r^r \to \|\varphi\|_r^r$ as $n \to \infty$, thus proving the theorem. □

11.3 Limit Theorems About Edgeworth Expansions

As is well-known, in case of the finite 3-rd absolute moment $\beta_3 = \mathbb{E}\,|X|^3$, and assuming the smoothness condition (11.2), the local limit theorems (11.6)–(11.7) can be sharpened to

$$\sup_x\, (1+|x|^3)\,|p_n(x) - \varphi(x)| = o\Big(\frac{1}{\sqrt{n}}\Big) \qquad (n \to \infty). \tag{11.10}$$

Here, the rate cannot be improved in general. However, under higher order moment assumptions, the limit normal density may slightly be modified, which leads to the sharpening of the right-hand side of (11.10). Namely, if $\beta_m = \mathbb{E}\,|X|^m$ is finite for an integer $m \ge 2$, one may introduce the cumulants

$$\gamma_k = i^{-k}\,(\log f)^{(k)}(0), \qquad k = 1,\dots,m.$$

They represent certain polynomials in the moments $\alpha_i = \mathbb{E}X^i$ up to order k, namely,

$$\gamma_k = k! \sum (-1)^{j-1}\,(j-1)!\,\frac{1}{r_1!\dots r_k!}\,\Big(\frac{\alpha_1}{1!}\Big)^{r_1}\dots\Big(\frac{\alpha_k}{k!}\Big)^{r_k},$$

where $j = r_1 + \dots + r_k$ and where the summation is running over all tuples $(r_1,\dots,r_k)$ of non-negative integers such that $r_1 + 2r_2 + \dots + kr_k = k$.

For example, with our moment assumptions $\mathbb{E}X = 0$, $\mathrm{Var}(X) = 1$, we have $\gamma_1 = 0$, $\gamma_2 = 1$, $\gamma_3 = \alpha_3$, $\gamma_4 = \alpha_4 - 3$.

Definition 11.3.1 An Edgeworth correction of the standard normal law of order m for the distribution of Z_n is a finite signed measure ν_m with density

$$\varphi_m(x) = \varphi(x) + \varphi(x) \sum_{k=1}^{m-2} Q_k(x)\, n^{-k/2}, \tag{11.11}$$

where

$$Q_k(x) = \sum \frac{1}{r_1! \dots r_k!} \Big(\frac{\gamma_3}{3!}\Big)^{r_1} \dots \Big(\frac{\gamma_{k+2}}{(k+2)!}\Big)^{r_k} H_{k+2j}(x). \tag{11.12}$$

Here, the summation is running over all collections of non-negative integers $r_1, \dots, r_k$ such that $r_1 + 2r_2 + \dots + kr_k = k$, with notation $j = r_1 + \dots + r_k$.

As usual, H_k denotes the Chebyshev-Hermite polynomial of degree k with leading term x^k. The polynomial Q_k in (11.11) has degree at most $3(m-2)$ in the variable x. The index m for φ_m indicates that the cumulants up to γ_m participate in the construction. The sum in (11.11) may also be viewed as a polynomial in $1/\sqrt{n}$ of degree at most $m-2$.

For example, $\varphi_2 = \varphi$, and there are no terms in the sum (11.11). For $m = 3, 4, 5, 6$, in (11.12) we correspondingly have

$$\begin{aligned}
Q_1(x) &= \frac{\gamma_3}{3!}\, H_3(x), \\
Q_2(x) &= \frac{\gamma_3^2}{2!\,3!^2}\, H_6(x) + \frac{\gamma_4}{4!}\, H_4(x), \\
Q_3(x) &= \frac{\gamma_3^3}{3!^4}\, H_9(x) + \frac{\gamma_3\gamma_4}{3!\,4!}\, H_7(x) + \frac{\gamma_5}{5!}\, H_5(x), \\
Q_4(x) &= \frac{\gamma_3^4}{4!\,3!^4}\, H_{12}(x) + \frac{\gamma_3^2\gamma_4}{2!\,3!^2\,4!}\, H_{10}(x) + \frac{\gamma_3\gamma_5}{3!\,5!}\, H_8(x) + \frac{\gamma_4^2}{2!\,4!^2}\, H_8(x) + \frac{\gamma_6}{6!}\, H_6(x).
\end{aligned}$$

Moreover, if the first $m-1$ moments of X coincide with those of $Z \sim N(0,1)$, then the first $m-1$ cumulants of X are vanishing, and (11.11) is simplified to

$$\varphi_m(x) = \varphi(x)\Big(1 + \frac{\gamma_m}{m!}\, H_m(x)\, n^{-\frac{m-2}{2}}\Big), \qquad \gamma_m = \mathbb{E}X^m - \mathbb{E}Z^m. \tag{11.13}$$

The following observation, generalizing and refining the non-uniform local limit theorems (11.7) and (11.10), is due to Petrov [14], cf. also [3, 15]. From now on, we always assume that the smoothing condition (11.2) is fulfilled.

Lemma 11.3.2 *If $\beta_m < \infty$ for an integer $m \geq 2$, then as $n \to \infty$*

$$\sup_x\, (1 + |x|^m)\, |p_n(x) - \varphi_m(x)| \,=\, o\big(n^{-\frac{m-2}{2}}\big). \tag{11.14}$$

Without the polynomial weight $1 + |x|^m$, a similar result was earlier obtained by Gnedenko. However, in some applications the appearance of this weight turns out to be crucial.

If $m \geq 3$, one may also take φ_{m-1} as an approximation of p_n, and then (11.14) together with Definition 11.3.1 imply that

$$\sup_x \, (1 + |x|^m) \, |p_n(x) - \varphi_{m-1}(x)| \, = \, O\big(n^{-\frac{m-2}{2}}\big). \tag{11.15}$$

A further generalization was given in [7] to employ the case of fractional moments.

Lemma 11.3.3 *Let $\beta_s < \infty$ for some real $s \geq 2$, and $m = [s]$. Then uniformly over all x, as $n \to \infty$,*

$$(1+|x|^s)\,(p_n(x)-\varphi_m(x)) \, = \, o\big(n^{-\frac{s-2}{2}}\big)+(1+|x|^{s-m})\,\Big(O\big(n^{-\frac{m-1}{2}}\big)+o\big(n^{-(s-2)}\big)\Big).$$

In particular, for some constant $\alpha > 0$ depending on s,

$$\sup_{|x| \leq n^{\alpha}} \, (1 + |x|^s) \, |p_n(x) - \varphi_m(x)| \, = \, o\big(n^{-\frac{s-2}{2}}\big). \tag{11.16}$$

Thus, (11.16) extends (11.14) when taking the supremum over relatively large interval.

There are also similar results about the distribution functions $F_n(x) = \mathbb{P}\{Z_n \leq x\}$, which may be approximated by

$$\Phi_m(x) \, = \, \nu_m((-\infty, x]) \, = \, \int_{-\infty}^{x} \varphi_m(y)\, dy \, = \, \Phi(x) - \varphi(x) \sum_{k=1}^{m-2} R_k(x)\, n^{-k/2}, \tag{11.17}$$

where

$$R_k(x) = \sum \frac{1}{r_1! \dots r_k!} \Big(\frac{\gamma_3}{3!}\Big)^{r_1} \dots \Big(\frac{\gamma_{k+2}}{(k+2)!}\Big)^{r_k} \, H_{k+2j-1}(x)$$

with summation as in Definition 11.3.1. The next result is due to Osipov and Petrov [13].

Lemma 11.3.4 *Suppose that $\beta_s < \infty$ for some real $s \geq 2$, and let $m = [s]$. Then, as $n \to \infty$,*

$$\sup_x \, (1 + |x|^s) \, |F_n(x) - \Phi_m(x)| \, = \, o\big(n^{-\frac{s-2}{2}}\big).$$

In particular, when $s = m \geq 3$ is integer, we have

$$\sup_x \, (1 + |x|^s) \, |F_n(x) - \Phi_{m-1}(x)| \, = \, O\big(n^{-\frac{s-2}{2}}\big).$$

This statement holds under the weaker assumption in comparison with (11.2): nothing should be required in case $2 \le s < 3$, while for $s \ge 3$ the Cramér condition is sufficient.

Remark 11.3.5 Since the densities p_n can properly be approximated by the functions φ_m, it makes sense to isolate the leading term in the sum (11.11), by rewriting the definition as

$$\varphi_m(x) = \varphi(x) + \varphi(x) \frac{\gamma_{k+2}}{(k+2)!} H_{k+2}(x)\, n^{-k/2} + \varphi(x) \sum_{j=k+1}^{m-2} Q_j(x)\, n^{-j/2} \tag{11.18}$$

for some unique $1 \le k \le m-2$. The value of k is the maximal one in the interval $[1, m-2]$ such that $\gamma_3 = \cdots = \gamma_{k+1} = 0$, which means that the first moments of X up to order $k+1$ coincide with those of $Z \sim N(0,1)$. In this case, necessarily $\gamma_{k+2} = \mathbb{E}X^{k+2} - \mathbb{E}Z^{k+2}$.

Of course, if $m = 2$, there are no terms on the right-hand side of (11.18) except for φ.

11.4 Approximation for L^r-Norm of Densities p_n

Lemmas 11.3.2–11.3.4 can be applied to explore an asymptotic behavior of the functionals

$$I(p) = \|p\|_r^r = \int_{-\infty}^{\infty} p(x)^r\, dx \qquad (r > 1)$$

with $p = p_n$. Since the densities p_n are approximated by φ_m, we may expect that $I(p_n) \sim I(\varphi_m)$ for large n. However, φ_m do not need to be positive on the whole real line, and it is more natural to consider the integrals

$$I_T(p) = \int_{|x| \le T} p(x)^r\, dx, \qquad T > 0,$$

over relatively long intervals. Actually, one may take $T = T_n = \sqrt{(s-2)\log n}$ $(s > 2)$. We have with some constants depending on the first m absolute moments of X that

$$\sum_{k=1}^{m-2} |Q_k(x)|\, n^{-k/2} \le C\,(1+|x|)^{3(m-2)} \frac{1}{\sqrt{n}} \le C' \,\frac{(\log n)^{3(m-2)/2}}{\sqrt{n}} \le \frac{1}{2}, \qquad |x| \le T_n,$$

for all n large enough in the last inequality. Hence, by Definition 11.3.1, for all n large enough,

$$|\varphi_m(x) - \varphi(x)| \leq \frac{1}{2}\,\varphi(x), \qquad |x| \leq T_n, \tag{11.19}$$

so φ_m is positive on $[-T_n, T_n]$. On these intervals and for large n, consider the functions

$$\varepsilon_n(x) = \frac{p_n(x) - \varphi_m(x)}{\varphi_m(x)}.$$

By (11.16) and (11.19), for $|x| \leq T_n$, we have

$$|\varepsilon_n(x)| \leq 2\delta_n \frac{n^{-\frac{s-2}{2}}}{\varphi(x)} \leq 2\sqrt{2\pi}\,\delta_n,$$

for some positive sequence $\delta_n \to 0$. Thus, for large n, $p_n(x) = \varphi_m(x)(1 + \varepsilon_n(x))$ with $|\varepsilon_n(x)| \leq \frac{1}{2}$. Hence, by Taylor's formula, and using (11.19) together with the non-uniform bound (11.16), we get

$$\begin{aligned} |\,p_n(x)^r - \varphi_m(x)^r\,| &\leq c\,\varphi(x)^r\,|\varepsilon_n(x)| \\ &\leq 2c\,\varphi(x)^{r-1}\,|p_n(x) - \varphi_m(x)| \,\leq\, \delta_n\,\frac{\varphi(x)^{r-1}}{1+|x|^s}\,n^{-\frac{s-2}{2}} \end{aligned}$$

with some constant c which does not depend on x and $n \geq n_0$ and some positive sequence $\delta_n \to 0$. After integration over $[-T_n, T_n]$, this gives

$$I_T(p_n) = I_T(\varphi_m) + o(n^{-\frac{s-2}{2}}). \tag{11.20}$$

In case $s = m \geq 3$ is integer, by a similar argument based on (11.15), we also have

$$I_T(p_n) = I_T(\varphi_{m-1}) + O(n^{-\frac{s-2}{2}}). \tag{11.21}$$

The remaining part of the integral,

$$J_T(p) = \int_{|x|>T} p(x)^r\,dx,$$

can be shown to be sufficiently small for $p = p_n$ on the basis of Lemma 11.3.4. Indeed, first

$$\mathbb{P}\{|Z| > T_n\} \leq \frac{1}{T_n}\,e^{-T_n^2/2} = o\big(n^{-\frac{s-2}{2}}\big), \qquad Z \sim N(0,1).$$

On the other hand, by Definition 11.3.1, using polynomial bounds $|Q_k(x)| \leq c_k\,(1+|x|^N)$ with $N = 3(m-2)$ and some constants c_k which do not depend on x, we have

$$|\varphi_m(x)| \leq \varphi(x) + \frac{c}{\sqrt{n}}\,(1+|x|^N)\,\varphi(x)$$

with some c independent of x and n. In addition,

$$\int_{|x|>T_n} |x|^N\,\varphi(x)\,dx \ \leq\ c'_N\,(1+T_n^N)\,e^{-T_n^2/2} \ \leq\ c''_N\,\log(n)^{\frac{N}{2}}\,n^{-\frac{s-2}{2}}$$

with constants c'_N and c''_N independent of n. This gives

$$\begin{aligned}
\big|\nu_m\{|x|>T_n\}\big| &\leq \int_{|x|>T_n} |\varphi_m(x)|\,dx \\
&\leq \int_{|x|>T_n} \varphi(x)\,dx + \frac{c}{\sqrt{n}}\int_{|x|>T_n} (1+|x|^N)\,\varphi(x)\,dx \\
&\leq \mathbb{P}\{|Z|>T_n\} + \frac{c''_N}{\sqrt{n}}\,\log(n)^{\frac{N}{2}}\,n^{-\frac{(s-2)}{2}},
\end{aligned}$$

and thus

$$\big|\nu_m\{|x|>T_n\}\big| \ =\ o\big(n^{-\frac{s-2}{2}}\big).$$

Since we assume the smoothness condition (11.2), the densities p_n are uniformly bounded by some constant M for all $n \geq n_0$. Therefore, by Lemma 11.3.4, for all n large enough,

$$\begin{aligned}
J_T(p_n) &\leq M^{r-1}\int_{|x|>T_n} p_n(x)\,dx \ =\ M^{r-1}\,\mathbb{P}\{|Z_n|>T_n\} \\
&\leq M^{r-1}\,\big|\nu_m\{x:|x|>T_n\}\big| + T_n^{-s}\,o\big(n^{-\frac{s-2}{2}}\big) \ =\ o\big(n^{-\frac{s-2}{2}}\big).
\end{aligned}$$

Combining this relation with (11.20) and (11.21), we arrive at:

Lemma 11.4.1 *Suppose that $\beta_s < \infty$ for $s \geq 2$. Then for all n large enough, Z_n have bounded densities p_n. Moreover, for any $r > 1$, as $n \to \infty$,*

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \ =\ \int_{-T_n}^{T_n} \varphi_m(x)^r\,dx + o\big(n^{-\frac{s-2}{2}}\big), \qquad m = [s], \tag{11.22}$$

where $T_n = \sqrt{(s-2)\log n}$. In particular, if $s = m \geq 3$ is integer, we also have

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \ =\ \int_{-T_n}^{T_n} \varphi_{m-1}(x)^r\,dx + O\big(n^{-\frac{s-2}{2}}\big). \tag{11.23}$$

11.5 Truncated L^r-Norm of Approximating Densities φ_m

Let us now find an explicit expression for the second integral in (11.22), by applying the Edgeworth approximation

$$\varphi_m(x) = \varphi(x)\Big(1 + \sum_{k=1}^{m-2} Q_k(x)\, n^{-k/2}\Big), \qquad m = [s]. \tag{11.24}$$

In the case $2 < s < 3$, when $\varphi_m = \varphi_2 = \varphi$, one may extend the integration in (11.22) to the whole real line at the expense of the error

$$\int_{|x|>T_n} \varphi(x)^r\, dx \ < \int_{|x|>T_n} \varphi(x)\, dx \ = \ \mathbb{P}\{|Z| > T_n\} \ = \ o\big(n^{-\frac{s-2}{2}}\big),$$

where $T_n = \sqrt{(s-2)\log n}$ as before. Hence, (11.22) yields

$$\int_{-\infty}^{\infty} p_n(x)^r\, dx \ = \int_{-\infty}^{\infty} \varphi(x)^r\, dx + o\big(n^{-\frac{s-2}{2}}\big), \qquad 2 < s < 3. \tag{11.25}$$

This assertion remains to hold for $s = 2$ as well (Theorem 11.1.1).

Next, assume that $s \geq 3$. As we know, when n is large enough, $\varphi_m(x)$ is positive for $|x| \leq T_n$, so the second integral in (11.22) makes sense, cf. (11.19). Moreover, in order to raise $\varphi_m(x)$ to the power r on the basis of (11.24), one may apply the Taylor expansion

$$(1+\varepsilon)^r \ = \ 1 + \sum_{k=1}^{N} \frac{(r)_k}{k!}\, \varepsilon^k + O(\varepsilon^{N+1}), \qquad N = 1, 2, \dots, \ \ (\varepsilon \to 0),$$

where the constant in O depends on N only, as long as $|\varepsilon| \leq \frac{1}{2}$. Here we used the standard notation $(r)_k = r(r-1)\dots(r-k+1)$, with convention $(r)_0 = 1$ to be used later on. Choosing

$$\varepsilon = \sum_{k=1}^{m-2} Q_k(x)\, n^{-k/2}, \qquad |x| \leq T_n,$$

for all n large enough the above Taylor expansion is thus valid. Hence, uniformly over all $x \in [-T_n, T_n]$, as $n \to \infty$,

$$(1+\varepsilon)^r \ = \ 1 + \sum_{k=1}^{N} \frac{(r)_k}{k!}\, \varepsilon^k + \varepsilon_n(x) \tag{11.26}$$

with

$$\varepsilon_n(x) = O\Big((1+|x|)^{3(m-2)(N+1)}\, n^{-(N+1)/2}\Big).$$

Furthermore, by the polynomial formula,

$$\varepsilon^k = \sum \frac{k!}{k_1! \dots k_{m-2}!}\, Q_1^{k_1}(x) \dots Q_{m-2}^{k_{m-2}}(x)\, n^{-\frac{1}{2}\,(k_1+2k_2+\dots+(m-2)\,k_{m-2})},$$

where the summation is running over all non-negative integers $k_1, \dots, k_{m-2}$ such that $k_1 + \dots + k_{m-2} = k$. Inserting this in (11.26) and recalling (11.24), we can represent $\varphi_m(x)^r$ as

$$\varphi(x)^r \sum \frac{(r)_{k_1+\dots+k_{m-2}}}{k_1! \dots k_{m-2}!}\, Q_1^{k_1}(x) \dots Q_{m-2}^{k_{m-2}}(x)\, n^{-\frac{1}{2}\,(k_1+2k_2+\dots+(m-2)\,k_{m-2})} + \varphi(x)^r\, \varepsilon_n(x)$$

with summation over all non-negative integers $k_1, \dots, k_{m-2}$ such that $k_1 + \dots + k_{m-2} \leq N$. One may now note that

$$\int_{-T_n}^{T_n} \varphi(x)^r\, \varepsilon_n(x)\, dx = O\big(n^{-\frac{N+1}{2}}\big),$$

where the constant in O depends on N only, as long as n is large enough.

Let us then choose $N = m-2$. Integrating the above expression for $\varphi_m(x)^r$ over the interval $[-T_n, T_n]$, we can represent $\int_{-T_n}^{T_n} \varphi_m(x)^r\, dx$ as

$$\sum \frac{(r)_{k_1+\dots+k_{m-2}}}{k_1! \dots k_{m-2}!} \int_{-T_n}^{T_n} \varphi(x)^r\, Q_1^{k_1}(x) \dots Q_{m-2}^{k_{m-2}}(x)\, dx\, \frac{1}{n^{\frac{1}{2}\,(k_1+2k_2+\dots+(m-2)\,k_{m-2})}}$$

at the expense of an error $O(n^{-\frac{m-1}{2}})$. Moreover, using the property

$$\int_{|x|\geq T_n} x^N \varphi(x)^r\, dx = o(n^{-\frac{s-2}{2}}),$$

the above integration may be extended to the whole real line. Hence, $\int_{-T_n}^{T_n} \varphi_m(x)^r\, dx$ is represented as

$$\sum \frac{(r)_{k_1+\dots+k_{m-2}}}{k_1! \dots k_{m-2}!} \int_{-\infty}^{\infty} \varphi(x)^r\, Q_1^{k_1}(x) \dots Q_{m-2}^{k_{m-2}}(x)\, dx\, \frac{1}{n^{\frac{1}{2}\,(k_1+2k_2+\dots+(m-2)\,k_{m-2})}} + o\big(n^{-\frac{s-2}{2}}\big).$$

Here, it is sufficient to keep only the powers of $1/n$ not exceeding $(m-2)/2$. But in that case, for any fixed value of

$$j = k_1 + 2k_2 + \dots + (m-2)\,k_{m-2},$$

the constraint $j \le m-2$ implies that $k_{j+1} = \dots = k_{m-2} = 0$. That is, we only need to consider the collections $k_1, \dots, k_j$ of length j. Thus, the above representation is simplified to

$$\int_{-T_n}^{T_n} \varphi_m(x)^r \, dx = \int_{-\infty}^{\infty} \varphi(x)^r \, dx$$
$$+ \sum \frac{(r)_{k_1+\dots+k_j}}{k_1! \dots k_j!} \int_{-\infty}^{\infty} \varphi(x)^r \, Q_1^{k_1}(x) \dots Q_j^{k_j}(x) \, dx \, n^{-j/2} + o\big(n^{-\frac{s-2}{2}}\big) \quad (11.27)$$

with summation over all $j = 1, \dots, m-2$ and over all non-negative integers $k_1, \dots, k_j$ such that $k_1 + 2k_2 + \dots + j\,k_j = j$.

As the last simplifying step, we note that $Q_{2k-1}(x)$ represents a linear combination of the polynomials $H_{2i-1}(x)$ and has a leading term $x^{3(2k-1)}$ up to a constant. In particular, it is an odd function. On the other hand, $Q_{2k}(x)$ represents a linear combination of $H_{2i}(x)$'s and has a leading term x^{6k}, so it is an even function. It follows that any function of the form

$$Q = Q_1^{k_1}(x) \dots Q_j^{k_j}(x) \qquad (k_1 + 2k_2 + \dots + j\,k_j = j) \quad (11.28)$$

is either odd or even, depending on whether j is odd or even. Indeed, for polynomials of the class 1, defined by

$$P(x) = c_0 + c_2 x^2 + \dots + c_{2N} x^{2N},$$

let us put $\mathrm{Ev}(P) = 2N \pmod 2 = 0$, and for the class 2, defined by

$$P(x) = c_1 x + \dots + c_{2N-1}\, x^{2N-1},$$

let us put $\mathrm{Ev}(P) = 2N - 1 \pmod 2 = 1$. The products of such polynomials belong to one of the classes, and we have the property $\mathrm{Ev}(P_1 P_2) = (\mathrm{Ev}(P_1) + \mathrm{Ev}(P_2)) \pmod 2$. Therefore, using $\mathrm{Ev}(Q_i) = 3i \pmod 2 = i \pmod 2$ and the summation in the group $\mathbb{Z}_2$, we have

$$\begin{aligned}\mathrm{Ev}(Q) &= k_1\,\mathrm{Ev}(Q_1) + \dots + k_j\,\mathrm{Ev}(Q_j) \\ &= k_1 \cdot 1 \pmod 2 + \dots + k_j \cdot j \pmod 2 \;=\; (k_1 + \dots + j k_j) \pmod 2 \;=\; j \pmod 2.\end{aligned}$$

Thus, Q is an odd function in (11.28), as long as j is odd, and then the corresponding integral in (11.27) is vanishing. As a result, (11.22) and (11.27) yield the following asymptotic expansion, which also holds for $2 \le s < 3$, in view of (11.25).

Proposition 11.5.1 *Suppose that $\beta_s < \infty$ for $s \geq 2$. Then, with $m = [s]$, for any $r > 1$,*

$$\int_{-\infty}^{\infty} p_n(x)^r\, dx \;=\; \int_{-\infty}^{\infty} \varphi(x)^r\, dx \left(1 + \sum_{j=1}^{[\frac{m-2}{2}]} \frac{a_j}{n^j}\right) + o\big(n^{-\frac{s-2}{2}}\big) \tag{11.29}$$

with coefficients defined by

$$a_j \int_{-\infty}^{\infty} \varphi(x)^r\, dx \;=\; \sum \frac{(r)_{k_1+\cdots+k_{2j}}}{k_1!\dots k_{2j}!} \int_{-\infty}^{\infty} Q_1^{k_1}(x)\dots Q_{2j}^{k_{2j}}(x)\, \varphi(x)^r\, dx. \tag{11.30}$$

Here, the summation runs over all integers $k_1, \dots, k_{2j} \geq 0$ such that $k_1 + 2k_2 + \cdots + 2j\, k_{2j} = 2j$ with notation $(r)_k = r(r-1)\dots(r-k+1)$.

It follows from Definition 11.3.1 that each polynomial Q_k is determined by the moments of X up to order $k+2$. Hence, each a_j in (11.30) is only determined by r and by the moments, hence, by the cumulants of X up to order $2j+2$. Moreover, $a_j = 0$ if these cumulants are vanishing.

11.6 The Case Where the First Cumulants Are Vanishing

For $2 \leq s < 4$, we necessarily have $m \leq 3$, so that the sum in (11.29) has no term, and then

$$\int_{-\infty}^{\infty} p_n(x)^r\, dx \;=\; \int_{-\infty}^{\infty} \varphi(x)^r\, dx + o\big(n^{-\frac{s-2}{2}}\big). \tag{11.31}$$

In the more interesting case $s \geq 4$, the leading term in the Edgeworth expansion (11.24) may be written explicitly, as was already done in the representation (11.18). It implies that, for some unique $1 \leq k \leq m-2$,

$$\varphi_m(x) \;=\; \varphi(x) + \varphi(x) \frac{\gamma_{k+2}}{(k+2)!}\, H_{k+2}(x)\, n^{-k/2} + C(x)\varphi(x)\,(1 + |x|^{3(m-2)})\, n^{-(k+1)/2} \tag{11.32}$$

with some function $C(x)$ bounded by a constant which does not depend on x and large $n \geq n_0$.

To study an asymptotic behavior of the truncated L^r-norm of φ_m, one may repeat computations of the previous section in this simple particular case, or alternatively, one may just refer to the general result described in Proposition 11.5.1. Indeed, (11.32) is equivalent to saying that the first moments of X up to order $k+1$ coincide with those of $Z \sim N(0,1)$ for some $1 \leq k \leq m-2$. Therefore, as emphasized after Proposition 11.5.1, $a_j = 0$ whenever $2j + 2 \le k + 1$, that is, $j \le \frac{k-1}{2}$. Then also $Q_j = 0$. In case $2j + 2 = k + 2$, that is, $j = k/2$ with even k, all terms in the sum (11.30) are vanishing, except (potentially) for the term corresponding to the collection with $k_1 = \dots = k_{2j-1} = 0$, $k_{2j} = 1$. Then the right-hand side of (11.30) becomes

$$r \int_{-\infty}^{\infty} Q_{2j}(x)\, \varphi(x)^r\, dx = r \int_{-\infty}^{\infty} Q_k(x)\, \varphi(x)^r\, dx = r\, \frac{\gamma_{k+2}}{(k+2)!} \int_{-\infty}^{\infty} H_{k+2}(x)\, \varphi(x)^r\, dx,$$

and hence (11.29) yields

$$\int_{-\infty}^{\infty} p_n(x)^r\, dx \;=\; \int_{-\infty}^{\infty} \varphi(x)^r\, dx + A n^{-k/2} + O(n^{-\frac{k+1}{2}}) + o\big(n^{-\frac{s-2}{2}}\big), \tag{11.33}$$

where

$$A = r\, \frac{\gamma_{k+2}}{(k+2)!} \int_{-\infty}^{\infty} H_{k+2}(x)\, \varphi(x)^r\, dx, \qquad \gamma_{k+2} = \mathbb{E}X^{k+2} - \mathbb{E}Z^{k+2}.$$

In particular, $A = 0$ for odd k, since then the Chebyshev-Hermite polynomial $H_{k+2}(x)$ is odd.

To proceed, we focus on the integrals $I(k,r) = \int_{-\infty}^{\infty} H_k(x)\, \varphi(x)^r\, dx$ with even k.

Lemma 11.6.1 *For any* $k = 1, 2, \dots,$

$$I(2k, r) \;=\; \frac{(2k-1)!!}{r^{\frac{2k+1}{2}}\, (2\pi)^{\frac{r-1}{2}}}\, (1-r)^k. \tag{11.34}$$

Proof The k-th Chebyshev-Hermite polynomial

$$H_k(x) \;=\; (-1)^k \left(e^{-x^2/2}\right)^{(k)} e^{x^2/2} \;=\; \mathbb{E}\,(x + iZ)^k, \qquad Z \sim N(0,1), \tag{11.35}$$

has generating function

$$\sum_{k=0}^{\infty} H_k(x)\, \frac{z^k}{k!} = e^{xz - z^2/2}, \qquad z \in \mathbb{C},$$

from which one can find the generating function for the sequence $c_k = I(k,r)$. Namely,

$$\sum_{k=0}^{\infty} c_k\, \frac{z^k}{k!} \;=\; \int_{-\infty}^{\infty} e^{xz - z^2/2}\, \varphi(x)^r\, dx \;=\; \frac{1}{(2\pi)^{\frac{r-1}{2}} \sqrt{r}}\, e^{-\frac{1}{2}(1-\frac{1}{r})\, z^2}.$$

Differentiating this equality $2k$ times and applying the definition (11.35), we arrive at

$$c_{2k} = \frac{1}{(2\pi)^{\frac{r-1}{2}}\sqrt{r}} \left(1 - \frac{1}{r}\right)^k H_{2k}(0).$$

It remains to apply the equality (11.35), which gives $H_{2k}(0) = (-1)^k\, \mathbb{E}Z^{2k} = (-1)^k\,(2k-1)!!$ □

For the first three even values $k = 2, 4, 6$, we thus have

$$I(2,r) = -\frac{1}{r^{3/2}\,(2\pi)^{\frac{r-1}{2}}}\,(r-1), \qquad I(4,r) = \frac{3}{r^{5/2}\,(2\pi)^{\frac{r-1}{2}}}\,(r-1)^2,$$
$$I(6,r) = -\frac{15}{r^{7/2}\,(2\pi)^{\frac{r-1}{2}}}\,(r-1)^3. \tag{11.36}$$

One may also evaluate the integrals $\int_{-\infty}^{\infty} H_k(x)^2\,\varphi(x)^r\,dx$. For example,

$$\int_{-\infty}^{\infty} H_3(x)^2\,\varphi(x)^r\,dx = \frac{1}{\sqrt{r}\,(2\pi)^{\frac{r-1}{2}}}\,\mathbb{E}\left(\left(\frac{Z}{\sqrt{r}}\right)^3 - 3\left(\frac{Z}{\sqrt{r}}\right)\right)^2 = \frac{3\,(5-6r+3r^2)}{r^{7/2}\,(2\pi)^{\frac{r-1}{2}}}. \tag{11.37}$$

Thus, the formula (11.34) may be used in the asymptotic representation (11.33). The particular case $k = [s]-2$ should be mentioned separately.

Corollary 11.6.2 *Suppose that $\mathbb{E}X^l = \mathbb{E}Z^l$ for $l = 1,\dots,m-1$ ($m \geq 3$), where $Z \sim N(0,1)$. If $\beta_s < \infty$ for some $s \in [m, m+1)$, then for all n large enough, Z_n have bounded densities p_n. Moreover,*

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx = \int_{-\infty}^{\infty} \varphi(x)^r\,dx + An^{-\frac{m-2}{2}} + o\big(n^{-\frac{s-2}{2}}\big) \tag{11.38}$$

with $A = 0$ in the case $m = 2k-1$ is odd, while in the case where $m = 2k$ is even, we have

$$A = \frac{\gamma_{2k}}{2^k k!}\,\frac{(1-r)^k}{(2\pi)^{\frac{r-1}{2}}\,r^{\frac{2k-1}{2}}}, \qquad \gamma_{2k} = \mathbb{E}X^{2k} - \mathbb{E}Z^{2k}.$$

If $\beta_s < \infty$ for $s = m+1$, then o-term in (11.38) may be replaced with O-term.

For example, if $\gamma_3 = \mathbb{E}X^3 = 0$, so that $m = 4$, $4 \leq s < 5$, we have

$$A = \frac{\gamma_4}{8}\,\frac{1}{(2\pi)^{\frac{r-1}{2}}}\,\frac{(1-r)^2}{r^{\frac{3}{2}}}, \qquad \gamma_4 = \mathbb{E}X^4 - \mathbb{E}Z^4 = \mathbb{E}X^4 - 3,$$

and (11.38) becomes

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \;=\; \int_{-\infty}^{\infty} \varphi(x)^r\,dx + An^{-1} + o\big(n^{-\frac{s-2}{2}}\big). \tag{11.39}$$

By (11.33), a similar formula remains to hold in the case $5 \le s < 6$, but then the o-term should be replaced with $O(n^{-3/2})$.

11.7 Moments of Order $4 \le s \le 8$

Returning to the general expansion (11.29) in Proposition 11.5.1 with coefficients a_j described in (11.30), let us now derive formulas similar to (11.39) for two regions of the values of s without additional assumptions on the first cumulants. To evaluate the integrals in that definition, we will use the formulas for the polynomials Q_j described in Sect. 11.3 for the indexes $j \le 4$.

If $4 \le s < 6$, the expansion (11.29) contains only one term, namely, we get

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \;=\; \int_{-\infty}^{\infty} \varphi(x)^r\,dx + \frac{a_1}{n}\int_{-\infty}^{\infty} \varphi(x)^r\,dx + o\big(n^{-\frac{s-2}{2}}\big) \tag{11.40}$$

with the coefficient for $j = 1$ in front of $1/n$, i.e.,

$$\begin{aligned} A_1 \;\equiv\; a_1 \int_{-\infty}^{\infty} \varphi(x)^r\,dx &= \frac{(r)_1}{1!}\int_{-\infty}^{\infty} Q_2(x)\,\varphi(x)^r\,dx + \frac{(r)_2}{2!}\int_{-\infty}^{\infty} Q_1^2(x)\,\varphi(x)^r\,dx \\ &= r\int_{-\infty}^{\infty}\Big(\frac{\gamma_4}{4!}\,H_4(x) + \frac{1}{2!}\Big(\frac{\gamma_3}{3!}\Big)^2 H_6(x)\Big)\,\varphi(x)^r\,dx \\ &\quad + \frac{r(r-1)}{2}\int_{-\infty}^{\infty}\Big(\frac{\gamma_3}{3!}\,H_3(x)\Big)^2\,\varphi(x)^r\,dx. \end{aligned}$$

Applying the formulas (11.36)–(11.37), we find that

$$\begin{aligned} A_1 &= r\,\frac{\gamma_3^2}{2!\,3!^2}\,I(6,r) + r\,\frac{\gamma_4}{4!}\,I(4,r) + \frac{r(r-1)}{2}\Big(\frac{\gamma_3}{3!}\Big)^2\int_{-\infty}^{\infty} H_3(x)^2\,\varphi(x)^r\,dx \\ &= -r\,\frac{\gamma_3^2}{72}\,\frac{15}{r^{7/2}\,(2\pi)^{\frac{r-1}{2}}}\,(r-1)^3 + r\,\frac{\gamma_4}{24}\,\frac{3}{r^{5/2}\,(2\pi)^{\frac{r-1}{2}}}\,(r-1)^2 + r(r-1)\,\frac{\gamma_3^2}{24}\,\frac{5-6r+3r^2}{r^{7/2}\,(2\pi)^{\frac{r-1}{2}}}. \end{aligned}$$

Equivalently,

$$(2\pi)^{\frac{r-1}{2}}\,\frac{r^{5/2}}{r-1}\,A_1 = -\frac{5}{24}\,(r-1)^2\,\gamma_3^2 + \frac{1}{8}\,r(r-1)\,\gamma_4 + \frac{1}{24}\,(5-6r+3r^2)\,\gamma_3^2.$$

Collecting the coefficients in front of γ_3^2, we arrive at the following refinement of (11.40).

Proposition 11.7.1 *Suppose that* $\beta_s < \infty$ *for* $4 \le s < 6$. *Then, for any* $r > 1$,

$$\int_{-\infty}^{\infty} p_n(x)^r \, dx \, = \int_{-\infty}^{\infty} \varphi(x)^r \, dx + A_1 n^{-1} + o\big(n^{-\frac{s-2}{2}}\big), \tag{11.41}$$

where the constant $A_1 = A_1(r)$ *is given by*

$$(2\pi)^{\frac{r-1}{2}} \frac{r^{3/2}}{r-1} \, A_1(r) \, = \, \frac{2-r}{12} \, \gamma_3^2 + \frac{r-1}{8} \, \gamma_4. \tag{11.42}$$

In the case $s = 6$, *the formula* (11.41) *remains valid with the remainder term* $O(n^{-2})$.

Note that $\lim_{r \to 1} \frac{A_1(r)}{r-1} = \frac{1}{12} \gamma_3^2$. Also, if $\gamma_3 = 0$, then (11.42) is simplified and defines exactly the constant A in the equality (11.39).

For the region $6 \le s < 8$, the sum in (11.29) contains two terms, proportional to $\frac{1}{n}$ and $\frac{1}{n^2}$. The coefficient a_1 is as before, while according to (11.30), we arrive at the following refinement.

Proposition 11.7.2 *Suppose that* $\beta_s < \infty$ *for* $6 \le s < 8$. *Then, for any* $r > 1$,

$$\int_{-\infty}^{\infty} p_n(x)^r \, dx \, = \int_{-\infty}^{\infty} \varphi(x)^r \, dx + A_1 n^{-1} + A_2 n^{-2} + o\big(n^{-\frac{s-2}{2}}\big), \tag{11.43}$$

where A_1 *is given in* (11.42) *and*

$$\begin{aligned} A_2 = r \int_{-\infty}^{\infty} Q_4(x) \, \varphi(x)^r \, dx + \frac{(r)_2}{2} \int_{-\infty}^{\infty} \big(Q_2^2(x) + 2\, Q_1(x) Q_3(x)\big) \, \varphi(x)^r \, dx \\ + \frac{(r)_3}{2} \int_{-\infty}^{\infty} Q_1^2(x) Q_2(x) \, \varphi(x)^r \, dx + \frac{(r)_4}{24} \int_{-\infty}^{\infty} Q_1^4(x) \, \varphi(x)^r \, dx. \end{aligned}$$

In the case $s = 8$, *the formula* (11.43) *remains valid with the remainder term* $O(n^{-3})$.

One can rewrite A_2 explicitly in terms of the cumulants of X, cf. [6]. In the case $\gamma_3 = 0$, a long expression for this constant is simplified to

$$A_2 \, = \, \frac{\gamma_6 \, r}{6!} \int_{-\infty}^{\infty} H_6(x) \, \varphi(x)^r \, dx + \frac{\gamma_4^2 \, r}{2! \, 4!^2} \int_{-\infty}^{\infty} H_8(x) \, \varphi(x)^r \, dx + \frac{\gamma_4^2 \, r(r-1)}{2! \, 4!^2} \int_{-\infty}^{\infty} H_4(x)^2 \, \varphi(x)^r \, dx.$$

11.8 Expansions for Rényi Entropies

Let us now reformulate the asymptotic results about the integrals $\int_{-\infty}^{\infty} p_n(x)^r\,dx$ in terms of the Rényi entropies and entropy powers

$$h_r(Z_n) \;=\; -\frac{1}{r-1}\,\log\int_{-\infty}^{\infty} p_n(x)^r\,dx, \qquad N_r(Z_n) \;=\; \Big(\int_{-\infty}^{\infty} p_n(x)^r\,dx\Big)^{-\frac{2}{r-1}}.$$

Since these functionals represent smooth functions of the L^r-norm, from Proposition 11.5.1 together with Taylor's formulas

$$\log(a+b+c) = \log a + a^{-1}b + O(b^2+|c|), \tag{11.44}$$
$$(a+b+c)^q = a^q + qa^{q-1}b + O(b^2+|c|),$$

holding with $a>0$, $q\neq 0$, and $b,c\to 0$, we immediately obtain:

Proposition 11.8.1 *Let $\mathbb{E}\,|X|^s < \infty$ for some $s\geq 2$, and $m=[s]$. Then, for any $r>1$,*

$$h_r(Z_n) = h_r(Z) + \sum_{j=1}^{[\frac{m-2}{2}]}\frac{b_j}{n^j} + o\big(n^{-\frac{s-2}{2}}\big), \tag{11.45}$$

$$N_r(Z_n) = N_r(Z)\left(1+\sum_{j=1}^{[\frac{m-2}{2}]}\frac{c_j}{n^j}\right) + o\big(n^{-\frac{s-2}{2}}\big), \tag{11.46}$$

with coefficients b_j and c_j that are determined by r and by the moments of X up to order $2j+2$.

Proof of Theorem 11.1.2 To evaluate the first coefficients in the expansions (11.45)–(11.46), we apply Taylor's formulas (11.44). For $q=-\frac{2}{r-1}$, the last equality in (11.44) reads

$$(a+b+c)^{-\frac{2}{r-1}} \;=\; a^{-\frac{2}{r-1}} - \frac{2}{r-1}\,a^{-\frac{r+1}{r-1}}\,b + O(b^2+|c|). \tag{11.47}$$

In particular (with $b=0$), the expansion of the form

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \;=\; \int_{-\infty}^{\infty}\varphi(x)^r\,dx + o\big(n^{-\frac{s-2}{2}}\big),$$

which corresponds to Proposition 11.5.1 to the region $2 < s < 4$, implies

$$\log \int_{-\infty}^{\infty} p_n(x)^r\,dx = \log \int_{-\infty}^{\infty} \varphi(x)^r\,dx + o\big(n^{-\frac{s-2}{2}}\big).$$

Equivalently, $h_r(Z_n) = h_r(Z) + o(n^{-\frac{s-2}{2}})$ or $N_r(Z_n) = N_r(Z) + o(n^{-\frac{s-2}{2}})$ for $Z \sim N(0,1)$.

More generally, applying (11.44)–(11.47) to the expansion

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \,=\, \int_{-\infty}^{\infty} \varphi(x)^r\,dx + A_1\,n^{-1} + o\big(n^{-\frac{s-2}{2}}\big),$$

corresponding to Proposition 11.7.1 with its region $4 \le s < 6$, we get

$$\log \int_{-\infty}^{\infty} p_n(x)^r\,dx \,=\, \log \int_{-\infty}^{\infty} \varphi(x)^r\,dx + A_1\,n^{-1}\bigg(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\bigg)^{-1} + o\big(n^{-\frac{s-2}{2}}\big),$$

and

$$\begin{aligned}\bigg(\int_{-\infty}^{\infty} p_n(x)^r\,dx\bigg)^{-\frac{2}{r-1}} = {} & \bigg(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\bigg)^{-\frac{2}{r-1}} \\ & - \frac{2A_1}{r-1}\,n^{-1}\bigg(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\bigg)^{-\frac{r+1}{r-1}} + o\big(n^{-\frac{s-2}{2}}\big).\end{aligned}$$

Thus,

$$h_r(Z_n) \,=\, h_r(Z) - \frac{A_1}{r-1}\,N_r(Z)^{\frac{r-1}{2}}\,n^{-1} + o(n^{-\frac{s-2}{2}}), \tag{11.48}$$

and (equivalently)

$$N_r(Z_n) \,=\, N_r(Z)\left[1 - \frac{2A_1}{r-1}\,N_r(Z)^{\frac{r-1}{2}}\,n^{-1}\right] + o(n^{-\frac{s-2}{2}}). \tag{11.49}$$

Recall that $A_1 = A_1(r)$ is determined by r and the cumulants $\gamma_3 = \mathbb{E}X^3$ and $\gamma_4 = \mathbb{E}X^4 - 3$. More precisely, according to the formula (11.42) of Proposition 11.7.1,

$$\frac{A_1}{r-1} \,=\, \frac{1}{(2\pi)^{\frac{r-1}{2}}\,r^{3/2}}\left[\frac{2-r}{12}\,\gamma_3^2 + \frac{r-1}{8}\,\gamma_4\right].$$

Since also

$$N_r(Z)^{\frac{r-1}{2}} = \bigg(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\bigg)^{-1} = (2\pi)^{\frac{r-1}{2}}\,r^{1/2},$$

the coefficients b_1 and c_1 in (11.45)–(11.46) in front of n^{-1} are simplified according to (11.48)–(11.49) as

$$b_1 = -\frac{A_1}{r-1}\,N_r(Z)^{\frac{r-1}{2}} = -\frac{1}{r}\left[\frac{2-r}{12}\,\gamma_3^2 + \frac{r-1}{8}\,\gamma_4\right], \qquad c_1 = 2b_1.$$

$\square$

Let us complement the expansions of Theorem 11.1.2 with similar assertions corresponding to the scenario from Corollary 11.6.2, where the first $m-1$ moments of X coincide with those of $Z \sim N(0,1)$, for some integer $m \geq 3$. If β_s is finite for $s \in [m, m+1)$, in that case we have an expansion of the form

$$\int_{-\infty}^{\infty} p_n(x)^r\,dx \;=\; \int_{-\infty}^{\infty} \varphi(x)^r\,dx + An^{-\frac{m-2}{2}} + o\big(n^{-\frac{s-2}{2}}\big).$$

Hence, by (11.44)–(11.47),

$$\begin{aligned}\log \int_{-\infty}^{\infty} p_n(x)^r\,dx &= \log \int_{-\infty}^{\infty} \varphi(x)^r\,dx \\ &\quad + A\,n^{-\frac{m-2}{2}}\left(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\right)^{-1} + O(n^{-(m-2)}) + o\big(n^{-\frac{s-2}{2}}\big),\end{aligned}$$

and

$$\begin{aligned}\left(\int_{-\infty}^{\infty} p_n(x)^r\,dx\right)^{-\frac{2}{r-1}} &= \left(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\right)^{-\frac{2}{r-1}} \\ &\quad -\frac{2A}{r-1}\,n^{-\frac{m-2}{2}}\left(\int_{-\infty}^{\infty} \varphi(x)^r\,dx\right)^{-\frac{r+1}{r-1}} + O(n^{-(m-2)}) + o\big(n^{-\frac{s-2}{2}}\big).\end{aligned}$$

Since $m-2 > \frac{s-2}{2}$, here O-term may be removed. In addition, as before, the last integral with its power can be written as $N_r(Z)^{\frac{r+1}{2}}$. Therefore, we obtain the asymptotic relations

$$h_r(Z_n) = h_r(Z) - \frac{A}{r-1}\,N_r(Z)^{\frac{r-1}{2}}\,n^{-\frac{m-2}{2}} + o(n^{-\frac{s-2}{2}})$$

and

$$N_r(Z_n) \;=\; N_r(Z)\left[1 - \frac{2A}{r-1}\,N_r(Z)^{\frac{r-1}{2}}\,n^{-\frac{m-2}{2}}\right] + o(n^{-\frac{s-2}{2}})$$

in full analogy with (11.48)–(11.49). The only difference is that we have a different formula for the constant $A = A(r)$. As stated in Corollary 11.6.2, here $A = 0$ in the case $m = 2k - 1$ is odd, while in the case $m = 2k$ is even, we have

$$A = \frac{\gamma_{2k}}{2^k k!} \, \frac{(1-r)^k}{(2\pi)^{\frac{r-1}{2}} r^{\frac{2k-1}{2}}}, \qquad \gamma_{2k} = \mathbb{E}X^{2k} - \mathbb{E}Z^{2k}.$$

Using again $N_r(Z)^{\frac{r-1}{2}} = (2\pi)^{\frac{r-1}{2}} r^{1/2}$, the coefficients b_{k-1} and c_{k-1} in (11.45)–(11.46) in front of $n^{-\frac{m-2}{2}} = n^{-(k-1)}$ are simplified to

$$b_{k-1} = -\frac{A}{r-1} \, N_r(Z)^{\frac{r-1}{2}} = \frac{\gamma_{2k}}{2^k k!} \, \frac{(1-r)^{k-1}}{r^{k-1}}, \qquad c_{k-1} = 2b_{k-1}.$$

Let us also remind that, if $\beta_s < \infty$ for $s = m+1$, then o-term may be replaced with $O(n^{-\frac{m-1}{2}})$. We are thus ready to make a corresponding statement.

Proposition 11.8.2 *Suppose that $\mathbb{E}X^l = \mathbb{E}Z^l$ for $l = 3, \dots, m-1$ ($m \geq 3$). If $\beta_s < \infty$ for some $s \in [m, m+1)$, then for any $r > 1$,*

$$h_r(Z_n) = h_r(Z) + bn^{-\frac{m-2}{2}} + o(n^{-\frac{s-2}{2}}),$$
$$N_r(Z_n) = N_r(Z)\left(1 + 2b\, n^{-\frac{m-2}{2}}\right) + o(n^{-\frac{s-2}{2}})$$

with constant $b = 0$ in the case $m = 2k - 1$ is odd, while in the case $m = 2k$ is even,

$$b = b_{k-1} = \frac{\gamma_{2k}}{2^k k!} \left(\frac{1}{r} - 1\right)^{k-1}, \qquad \gamma_{2k} = \mathbb{E}X^{2k} - \mathbb{E}Z^{2k}.$$

If $\beta_s < \infty$ for $s = m+1$, then o-term may be replaced with $O(n^{-\frac{m-1}{2}})$.

For example, if $\gamma_3 = \mathbb{E}X^3 = 0$, we return to the equality (11.4) from Theorem 11.1.2.

11.9 Comparison with the Entropic CLT: Monotonicity

Put

$$\Delta_n(r) = h_r(Z) - h_r(Z_n), \qquad \Delta_n = \Delta_n(1).$$

The latter quantity, which may also be written as $D(Z_n||Z) = \int_{-\infty}^{\infty} p_n(x) \, \log \frac{p_n(x)}{\varphi(x)} \, dx$, represents the Kullback-Leibler distance from the distribution of Z_n to the standard normal law (or, the relative entropy). As was mentioned, the sequence Δ_n is always non-negative and non-increasing. Moreover, the entropic CLT asserts that $\Delta_n \to 0$ as $n \to \infty$, as long as Δ_n is finite for some n (in general, it is a weaker condition in comparison with (11.2)). The basic references for these results are [1, 2, 12].

The rate of convergence of Δ_n to zero was studied in [8], and here we recall a few asymptotic results, assuming that $\Delta_n < \infty$ for some n, and that $\beta_s = \mathbb{E}\,|X|^s < \infty$ for a real number $s \geq 2$. Namely, we have

$$\Delta_n = o\bigg(\frac{1}{(n\log n)^{\frac{s-2}{2}}}\bigg), \qquad 2 \leq s < 4.$$

Modulo a logarithmic term, it is the same rate as for $\Delta_n(r)$ indicated in Theorem 11.1.2. Nevertheless, it is not yet clear, if one can similarly improve Theorem 11.1.2. On the other hand, for any prescribed $\eta > 1$, it may occur that, for all n large enough,

$$\Delta_n \geq \frac{c}{(n\log n)^{\frac{s-2}{2}}\,(\log n)^{\eta}}$$

with some constant $c = c(\eta, s) > 0$ depending on η and s only ([8], Theorem 11.1.3).

The range $s \geq 4$ is more interesting, since then one may control the speed of Δ_n. In particular,

$$\Delta_n = \frac{\gamma_3^2}{12}\,n^{-1} + o\bigg(\frac{1}{(n\log n)^{\frac{s-2}{2}}}\bigg), \qquad 4 \leq s < 6,$$

$$\Delta_n = \frac{\gamma_3^2}{12}\,n^{-1} + O\bigg(\frac{1}{(n\log n)^2}\bigg), \qquad s = 6.$$

Thus, if $\gamma_3 \neq 0$, then Δ_n is equivalent to a decreasing sequence, which decreases at rate n^{-1}. (Strictly speaking, this property does not imply the monotonicity itself.)

Let us compare this asymptotic with what is given in Theorem 11.1.2. Namely, for any $r > 1$, we have

$$\Delta_n(r) = B_1\,n^{-1} + o\big(n^{-\frac{s-2}{2}}\big), \qquad 4 \leq s < 6, \tag{11.50}$$

$$\Delta_n(r) = B_1\,n^{-1} + O\big(n^{-2}\big), \qquad s = 6, \tag{11.51}$$

where

$$B_1 = B_1(r) = -b = \frac{1}{4r}\bigg[\frac{2-r}{3}\,\gamma_3^2 + \frac{r-1}{2}\,\gamma_4\bigg].$$

We see that $B(r) \to \frac{1}{12}\,\gamma_3^2$ as $r \to 1$, so that we recover the main term in the asymptotic for Δ_n, and at the same rate modulo a logarithmic factor.

However, what can one say about the sign of $B_1(r)$ with fixed $r > 1$? First suppose that $\gamma_3 \neq 0$. When r is sufficiently close to 1, then $B_1(r) > 0$, so that $\Delta_n(r)$ is equivalent to a decreasing sequence like for $r = 1$. More precisely, this is true for all $r > 1$, whenever $\gamma_4 \geq \frac{2}{3}\gamma_3^2$. But, if $\gamma_4 < \frac{2}{3}\gamma_3^2$, then $B_1(r) < 0$ for all

$$r > r_0 = \frac{4\gamma_3^2 - 3\gamma_4}{2\gamma_3^2 - 3\gamma_4}.$$

Hence $\Delta_n(r)$ becomes to be equivalent to an increasing sequence. In that case, necessarily $h_r(Z_n) > h_r(Z)$ for all n large enough, which is impossible in the Shannon case $r = 1$. This shows that $\Delta_n(r)$ may not serve as distance!

If $\gamma_3 = 0$ (as in case of symmetric distributions), the constant is simplified to

$$B_1 = B_1(r) = \frac{r-1}{8r}\,\gamma_4, \qquad \gamma_4 = \mathbb{E}X^4 - 3,$$

and then the sign of B_1 coincides with the sign of γ_4. Both cases, $\gamma_4 > 0$ or $\gamma_4 < 0$, are typical, and one can make a similar conclusion as before, but for the whole range $r > 1$. Namely, if $\gamma_4 > 0$, then $\Delta_n(r)$ is equivalent to a decreasing sequence, which decreases at rate n^{-1}, and if $\gamma_4 < 0$, then $\Delta_n(r)$ is equivalent to an increasing sequence, which increases also at rate n^{-1}.

Proof of Theorem 11.1.3 in Case $r < \infty$ In order to make a more rigorous conclusion about the monotonicity of $\Delta_n(r)$ for large n, the expansions for Rényi entropy $h_r(Z_n)$ such as (11.50)–(11.51) are insufficient. We need to use more terms in the general Proposition 11.8.1 involving the quadratic terms b_2/n^2 and c_2/n^2. This is possible under stronger moment assumptions, corresponding to the range $6 \leq s < 8$. Indeed, in that case, Proposition 11.8.1 provides the expansion (11.5) in which the coefficient $b_1 = b$ is as before, and we also know that the coefficient b_2 is only determined by r and by the moments of X up to order 6. In fact, one may evaluate b_2 on the basis of equality (11.43) of Proposition 11.7.2, which specializes Proposition 11.5.1 to the range $6 \leq s < 8$. Since the formula for the coefficient $A_2 = A_2(r)$ is somewhat complicated, we will not go into tedious computations.

Now, from (11.5) it follows that

$$h_r(Z_{n+1}) - h_r(Z_n) \;=\; \frac{B_1}{n(n+1)} + o(n^{-2}),$$

which thus proves Theorem 11.1.3 in case of finite r. □

11.10 Maximum of Density (the Case $r = \infty$)

Recall that $N_\infty(X) = \|p\|_\infty^{-2}$ when a random variable X has density p. An expansion similar to the one of Proposition 11.5.1 can also be obtained for $\|p_n\|_\infty$ and hence for $N_\infty(Z_n)$. In order to deduce monotonicity, let us assume that $\beta_6 < \infty$.

From the non-uniform local limit theorem it follows that $\|p_n - \varphi_6\|_\infty = o(n^{-2})$ as $n \to \infty$, where φ_6 is the Edgeworth expansion of order 6. Hence

$$\|p_n\|_\infty = \|\varphi_6\|_\infty + o(n^{-2}). \tag{11.52}$$

Here

$$\varphi_6(x) = \varphi(x)\Big(1 + Q_1(x)\frac{1}{\sqrt{n}} + Q_2(x)\frac{1}{n} + Q_3(x)\frac{1}{n^{\frac{3}{2}}} + Q_4(x)\frac{1}{n^2}\Big),$$

where the polynomials $Q_k(x)$ are the same as in Sect. 11.3.

Let us find an asymptotic expansion for $\|\varphi_6\|_\infty$ (we refer to [6] for more computational details). Since $\varphi_6(x)$ is vanishing at infinity, there exists a point $x_6(n)$ such that $\|\varphi_6\|_\infty = |\varphi_6(x_6(n))|$. Since also the functions $\varphi(x)\ Q_k(x)$ are bounded, we have $|\varphi_6(x)| = O(\frac{1}{\sqrt{n}})$ uniformly in the region $|x| \geq \sqrt{\log n}$. On the other hand,

$$\varphi_6(0) = \varphi(0) + \varphi(0)\sum_{k=1}^{4} Q_k(0)\, n^{-\frac{k}{2}} \geq \frac{1}{2}\,\varphi(0)$$

for n large. Therefore, $\varphi_6(0) > |\varphi_6(x)|$ for all n large enough, as long as $|x| \geq \sqrt{\log n}$, and we conclude that

$$\|\varphi_6\|_\infty = \sup_{|x|\leq\sqrt{\log n}} |\varphi_6(x)| \qquad \text{and} \qquad |x_6(n)| \leq \sqrt{\log n}. \tag{11.53}$$

Since $x = x_6(n)$ is the point of local extremum, we have $\varphi_6'(x) = 0$, that is,

$$x = \frac{Q_1'(x) - xQ_1(x)}{\sqrt{n}} + \frac{Q_2'(x) - xQ_2(x)}{n} + \frac{Q_3'(x) - xQ_3(x)}{n^{\frac{3}{2}}} + \frac{Q_4'(x) - xQ_4(x)}{n^2}. \tag{11.54}$$

Using (11.53), we deduce from (11.54) that $x_6(n) = O\big(\frac{1}{\sqrt{n}}\,(\log n)^{\frac{13}{2}}\big)$ and hence $|x_6(n)| \leq 1$ for all n large enough. But then, from (11.54) again, $x_6(n) = O(\frac{1}{\sqrt{n}})$. For $x = x_6(n)$, we thus have

$$\frac{xQ_3(x)}{n^{\frac{3}{2}}} = O\big(n^{-5/2}\big), \qquad \frac{Q_4'(x)}{n^2} = O\big(n^{-5/2}\big), \qquad \frac{xQ_4(x)}{n^2} = O\big(n^{-5/2}\big),$$

and (11.54) is simplified to

$$x = \frac{Q_1'(x) - xQ_1(x)}{\sqrt{n}} + \frac{Q_2'(x) - xQ_2(x)}{n} + \frac{Q_3'(x)}{n^{\frac{3}{2}}} + O\big(n^{-5/2}\big).$$

The Chebyshev-Hermite polynomials satisfy the relation $H_k'(x) - xH_k(x) = -H_{k+1}(x)$, so

$$\begin{aligned} H_3'(x) - xH_3(x) = -H_4(x) &= -3 + 6x^2 - x^4 \\ H_4'(x) - xH_4(x) = -H_5(x) &= -15x + 10x^3 - x^5 \\ H_6'(x) - xH_6(x) = -H_7(x) &= 105\,x - 105\,x^3 + 21\,x^5 - x^7. \end{aligned}$$

Using these identities in the formulas for Q_k's, we easily find for $x = O(\frac{1}{\sqrt{n}})$ that

$$\begin{aligned} \frac{Q_1'(x) - xQ_1(x)}{\sqrt{n}} &= -\frac{\gamma_3}{2\sqrt{n}} + \gamma_3 \frac{x^2}{\sqrt{n}} + O\big(n^{-5/2}\big), \\ \frac{Q_2'(x) - xQ_2(x)}{n} &= \Big(\frac{105}{2!\,3!^2}\,\gamma_3^2 - \frac{15}{4!}\,\gamma_4\Big)\,\frac{x}{n} + O\big(n^{-5/2}\big), \\ \frac{Q_3'(x)}{n^{\frac{3}{2}}} &= \Big(\frac{945}{3!^4}\,\gamma_3^3 - \frac{105}{3!\,4!}\,\gamma_3\gamma_4 + \frac{15}{5!}\,\gamma_5\Big)\,\frac{1}{n^{\frac{3}{2}}} + O\big(n^{-5/2}\big). \end{aligned}$$

As a result,

$$\begin{aligned} x \;=\; x_6(n) = -\frac{\gamma_3}{2\sqrt{n}} + \gamma_3 \frac{x^2}{\sqrt{n}} + \Big(\frac{105}{2 \cdot 3!^2}\,\gamma_3^2 - \frac{15}{4!}\,\gamma_4\Big)\frac{x}{n} \\ + \Big(\frac{945}{3!^4}\,\gamma_3^3 - \frac{105}{3!4!}\,\gamma_3\gamma_4 + \frac{15}{5!}\,\gamma_5\Big)\frac{1}{n^{\frac{3}{2}}} + O\big(n^{-5/2}\big). \end{aligned} \tag{11.55}$$

One may use this asymptotic equation to find an expansion for $x_6(n)$ in powers of $1/\sqrt{n}$. Indeed, first we immediately obtain that

$$x = x_6(n) = -\frac{\gamma_3}{2\sqrt{n}} + O\big(n^{-\frac{3}{2}}\big),$$

implying

$$\frac{x^2}{\sqrt{n}} \;=\; \frac{\gamma_3^2}{4}\,\frac{1}{n^{\frac{3}{2}}} + O\big(n^{-5/2}\big), \qquad \frac{x}{n} \;=\; -\frac{\gamma_3}{2}\,\frac{1}{n^{\frac{3}{2}}} + O\big(n^{-5/2}\big).$$

Inserting the above to (11.55), we deduce that

$$x = x_6(n) = \frac{a_1}{\sqrt{n}} + \frac{a_2}{n^{\frac{3}{2}}} + O\big(n^{-5/2}\big)$$

with coefficients

$$a_1 = -\frac{1}{2}\,\gamma_3, \qquad a_2 = \frac{1}{4}\,\gamma_3^3 - \frac{5}{12}\,\gamma_3\gamma_4 + \frac{1}{8}\,\gamma_5.$$

In particular, $a_1 = a_2 = 0$ and therefore $x = x_6(n) = O\big(n^{-5/2}\big)$, as long as the distribution of X is symmetric about the origin (in which case $\gamma_3 = \gamma_5 = 0$).

Still in the general case, keeping these coefficients, we deduce for $x = x_6(n)$ that

$$x = \frac{1}{\sqrt{n}}\Big(a_1 + a_2\,\frac{1}{n}\Big) + O\big(n^{-5/2}\big), \quad x^2 = \frac{1}{n}\Big(a_1^2 + 2a_1a_2\,\frac{1}{n}\Big) + O\big(n^{-5/2}\big),$$

$$x^3 = \frac{1}{n^{\frac{3}{2}}}\,a_1^3 + O\big(n^{-5/2}\big), \quad x^4 = \frac{1}{n^2}\,a_1^4 + O\big(n^{-5/2}\big), \quad x^p = O\big(n^{-5/2}\big) \quad (p \geq 5).$$

Hence

$$\frac{Q_1(x)}{\sqrt{n}} = \frac{\gamma_3}{6\sqrt{n}}\,(x^3 - 3x) = \frac{\gamma_3^2}{4n} + \frac{b_1}{n^2} + O\big(n^{-5/2}\big), \qquad b_1 = \frac{\gamma_3}{3!}\,(a_1^3 - 3a_2).$$

Similarly,

$$\frac{Q_2(x)}{n} = \Big(\frac{3}{4!}\,\gamma_4 - \frac{15}{2!\cdot 3!^2}\,\gamma_3^2\Big)\,\frac{1}{n} + \frac{b_2}{n^2} + O\big(n^{-5/2}\big),$$

$$\frac{Q_3(x)}{n^{\frac{3}{2}}} = \frac{b_3}{n^2} + O\big(n^{-5/2}\big),$$

$$\frac{Q_4(x)}{n^2} = \frac{b_4}{n^2} + O\big(n^{-5/2}\big)$$

with

$$b_2 = \Big(\frac{45}{2\cdot 3!^2}\,\gamma_3^2 - \frac{6}{4!}\,\gamma_4\Big)\,a_1^2, \qquad b_3 = \Big(\frac{945}{3!^4}\,\gamma_3^3 - \frac{105}{3!4!}\,\gamma_3\gamma_4 + \frac{15}{5!}\,\gamma_5\Big)\,a_1,$$

and

$$b_4 = \frac{10\,395}{4!\cdot 3!^4}\gamma_3^4 - \frac{945}{2\cdot 3!^2 4!}\,\gamma_3^2\gamma_4 + \frac{105}{3!\cdot 5!}\,\gamma_3\gamma_5 + \frac{105}{2\cdot 4!^2}\,\gamma_4^2 - \frac{15}{6!}\,\gamma_6.$$

Note that in the case of symmetric distributions, $b_1 = b_2 = b_3 = 0$, while $b_4 = \frac{105}{2\cdot 4!^2}\,\gamma_4^2 - \frac{15}{6!}\,\gamma_6$.

Now, as $x \to 0$,

$$\frac{\varphi(x)}{\|\varphi\|_\infty} = 1 - \frac{1}{2}\,x^2 + \frac{1}{8}\,x^4 + O(x^6),$$

and recall that, for $x = x_6(n)$, we have $x^2 = \frac{1}{n}\,(a_1^2 + 2a_1a_2\,\frac{1}{n}) + O(n^{-5/2})$ and $x^4 = \frac{1}{n^2}\,a_1^4 + O(n^{-5/2})$. Thus,

$$\frac{\varphi(x)}{\|\varphi\|_\infty} = 1 - \frac{a_1^2}{2n} + \Big(\frac{a_1^4}{8} - a_1a_2\Big)\frac{1}{n^2} + O\big(n^{-5/2}\big).$$

Therefore, denoting $b = b_1 + b_2 + b_3 + b_4$, we get

$$\begin{aligned}\frac{\|\varphi_6\|_\infty}{\|\varphi\|_\infty} = \frac{\varphi_6(x)}{\|\varphi\|_\infty} &= \frac{\varphi(x)}{\|\varphi\|_\infty}\Big(1 + \frac{Q_1(x)}{\sqrt{n}} + \frac{Q_2(x)}{n} + \frac{Q_3(x)}{n^{\frac{3}{2}}} + \frac{Q_4(x)}{n^2}\Big)\\ &= 1 + \Big(-\frac{1}{2}\,a_1^2 + \frac{1}{4}\,\gamma_3^2 + \frac{3}{4!}\,\gamma_4 - \frac{15}{2!\cdot 3!^2}\,\gamma_3^2\Big)\frac{1}{n}\\ &\quad + \bigg(b + \frac{1}{8}\,a_1^4 - a_1a_2 - \frac{1}{2}\,\Big(\frac{1}{4}\,\gamma_3^2 + \frac{3}{4!}\,\gamma_4 - \frac{15}{2!\cdot 3!^2}\,\gamma_3^2\Big)\,a_1^2\bigg)\,\frac{1}{n^2} + O\big(n^{-5/2}\big).\end{aligned}$$

Simplifying the term in front of $1/n$, we arrive at

$$\|\varphi_6\|_\infty = \|\varphi\|_\infty + \frac{\|\varphi\|_\infty}{n}\,A + \frac{\|\varphi\|_\infty}{n^2}\,B + O\big(n^{-5/2}\big),$$

where

$$A = \frac{1}{8}\Big(\gamma_4 - \frac{2}{3}\,\gamma_3^2\Big), \qquad B = b + \frac{1}{8}\,a_1^4 - a_1a_2 - \frac{1}{2}\,\Big(\frac{1}{4}\,\gamma_3^2 + \frac{3}{4!}\,\gamma_4 - \frac{15}{2!\cdot 3!^2}\,\gamma_3^2\Big)\,a_1^2. \tag{11.56}$$

Using our assumptions, let us summarize by recalling the assertion (11.52). We then get

$$\|p_n\|_\infty = \|\varphi\|_\infty\Big(1 + \frac{1}{n}\,A + \frac{1}{n^2}\,B\Big) + o\big(n^{-2}\big), \tag{11.57}$$

where A and B are as above with $a_1 = -\frac{1}{2}\,\gamma_3$ and $a_2 = \frac{1}{4}\,\gamma_3^3 - \frac{5}{12}\,\gamma_3\gamma_4 + \frac{1}{8}\,\gamma_5$.

One can now reformulate this result in terms of the Rényi entropy of index $r = \infty$. Since $N_\infty(Z_n) = \|p_n\|_\infty^{-2}$ and $N_\infty(Z) = \|\varphi\|_\infty^{-2}$ for $Z \sim N(0,1)$, the expansion (11.57) yields:

Proposition 11.10.1 *If β_6 is finite, then as $n \to \infty$,*

$$N_\infty(Z_n) = N_\infty(Z)\Big(1 - \frac{\widetilde{A}}{n} + \frac{\widetilde{B}}{n^2}\Big) + o\Big(\frac{1}{n^2}\Big)$$

with $\widetilde{A} = \frac{1}{4}\,(\gamma_4 - \frac{2}{3}\,\gamma_3^2)$, $\widetilde{B} = 3A^2 - 2B$, *where the constants* A *and* B *are given in* (11.56).

Proof of Theorem 11.1.3 in Case $r = \infty$ Denoting $\Delta_n = N_\infty(Z) - N_\infty(Z_n)$, from (11.57) we get $\Delta_{n+1} - \Delta_n = -\frac{\widetilde{A}}{n(n+1)} + o(\frac{1}{n^2})$. □

In the case $\gamma_3 = \gamma_5 = 0$, for example when X is symmetric, the coefficients in Proposition 11.10.1 are simplified. Indeed, recalling the formula for b_4 in such a case, we have

$$A = \frac{1}{8}\,\gamma_4, \qquad B = b_4 = \frac{105}{2 \cdot 4!^2}\,\gamma_4^2 - \frac{15}{6!}\,\gamma_6,$$

and therefore,

$$\widetilde{A} = \frac{1}{4}\,\gamma_4, \qquad \widetilde{B} = 3A^2 - 2B = \frac{1}{24}\,\gamma_6 - \frac{13}{96}\,\gamma_4^2.$$

As a consequence, the eventual monotonicity of $N_\infty(Z_n)$ can be deduced based on the sign of γ_4. However, if also $\gamma_4 = 0$, we need to look at the sign of γ_6.

Acknowledgements This research was partially supported by NSF grant DMS-1855575 and BSF grant 2016050. Also, this research was supported in part by the Walter S. Baer and Jeri Weiss CMI Postdoctoral Fellowship.

References

1. S. Artstein, K.M. Ball, F. Barthe, A. Naor, Solution of Shannon's problem on the monotonicity of entropy. J. Am. Math. Soc. **17**(4), 975–982 (2004)
2. A.R. Barron, Entropy and the central limit theorem. Ann. Probab. **14**(1), 336–342 (1986)
3. R.N. Bhattacharya, R. Ranga Rao, *Normal Approximation and Asymptotic Expansions* (Wiley, Hoboken, 1976). Also: Soc. for Industrial and Appl. Math., Philadelphia, 2010
4. S.G. Bobkov, G.P. Chistyakov, Entropy power inequality for the Rényi entropy. IEEE Trans. Inform. Theory **61**(2), 708–714 (2015)
5. S.G. Bobkov, A. Marsiglietti, Variants of the entropy power inequality. IEEE Trans. Inform. Theory **63**(12), 7747–7752 (2017)
6. S.G. Bobkov, A. Marsiglietti, Asymptotic behavior of Rényi entropy in the central limit theorem (extended version) (2018). Preprint. arXiv:1802.10212
7. S.G. Bobkov, G.P. Chistyakov, F. Götze, Non-uniform bounds in local limit theorems in case of fractional moments. I. Math. Methods Stat. **20**(3), 171–191 (2011); **20**(4), 269–287 (2011)
8. S.G. Bobkov, G.P. Chistyakov, F. Götze, Rate of convergence and Edgeworth-type expansion in the entropic central limit theorem. Ann. Probab. **41**(4), 2479–2512 (2013)
9. T.M. Cover, J.A. Thomas, *Elements of Information Theory*, 2nd edn. (Wiley-Interscience [Wiley], Hoboken, 2006) xxiv+748 pp.
10. A. Dembo, T.M. Cover, J.A. Thomas, Information-theoretic inequalities. IEEE Trans. Inform. Theory **37**(6), 1501–1518 (1991)

11. B.V. Gnedenko, A.N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*. Translated and annotated by K. L. Chung. With an Appendix by J. L. Doob (Addison-Wesley Publishing Company, Cambridge, 1954), ix+264 pp.
12. M. Madiman, A. Barron, Generalized entropy power inequalities and monotonicity properties of information. IEEE Trans. Inform. Theory **53**(7), 2317–2329 (2007)
13. L.V. Osipov, V.V. Petrov, On the estimation of the remainder term in the central limit theorem. Teor. Veroyatn. Primen. **12**, 322–329 (1967)
14. V.V. Petrov, Local limit theorems for sums of independent random variables (Russian). Teor. Verojatnost. Primenen. **9**, 343–352 (1964)
15. V.V. Petrov, *Sums of Independent Random Variables*. Translated from the Russian by A. A. Brown. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82 (Springer, New York, 1975), x+346 pp. Russian ed.: Moscow, Nauka, 1972, 414 pp.