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Refinements of Berry–Esseen Inequalities in Terms of Lyapunov Coefficients

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Abstract
We discuss some variants of the Berry–Esseen inequality in terms of Lyapunov coefficients which may provide sharp rates of normal approximation.

Keywords Central limit theorem · Berry–Esseen inequality · Fourier–Stieltjes transform

Mathematics Subject Classification Primary 60E · 60F

1 Introduction

Given independent random variables $(X_k)_{1 \leq k \leq n}$ with mean $\mathbb{E}X_k = 0$ and finite variances $\sigma_k^2 = \mathrm{Var}(X_k)$, denote by $F_n(x) = \mathbb{P}\{S_n \leq x\}$ the distribution function of the sum

$$S_n = X_1 + \cdots + X_n. \tag{1.1}$$

For normalization reason, we assume that $\mathbb{E}S_n^2 = \sigma_1^2 + \cdots + \sigma_n^2 = 1$.

It is well-known that, under the Lindeberg condition, F_n is close in the weak topology to the standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy, \quad x \in \mathbb{R}.$$

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In order to quantify the normal approximation, one often considers upper bounds for the Kolmogorov distance

$$\Delta_n = \sup_x |F_n(x) - \Phi(x)|$$

in terms of the Lyapunov coefficients

$$L_p = \sum_{k=1}^n \mathbb{E}\,|X_k|^p, \quad p > 2.$$

In the case of independent, identically distributed (i.i.d.) summands $X_k = \frac{1}{\sqrt{n}}\,\xi_k$ with finite absolute moment $\beta_p = \mathbb{E}\,|\xi_1|^p$, these quantities have a polynomial decay with respect to n:

$$L_p = \beta_p\, n^{-\frac{p-2}{2}}.$$

A basic fundamental relation in this direction is the classical Berry–Esseen inequality which indicates that

$$\Delta_n \le cL_3, \tag{1.2}$$

cf. e.g. [18]. Here and below, we use c to denote positive absolute constants which may vary from place to place (otherwise, we add parameters which these constants may depend on). In the i.i.d. scenario, (1.2) leads to the standard rate of normal approximation under the 3rd moment assumption,

$$\Delta_n \le c\frac{\beta_3}{\sqrt{n}}. \tag{1.3}$$

Much of the work has been done in order to polish the constants in these inequalities. The best known results in this respect are due to Shevtsova [22], who showed that one may take $c = 0.56$ in (1.2) and $c = 0.47$ in (1.3).

The Berry–Esseen inequality (1.2) may be sharpened as a non-uniform bound

$$\sup_x \left[(1 + |x|^3)\,|F_n(x) - \Phi(x)|\right] \le cL_3,$$

which is due to Nagaev [14] in the i.i.d. case and Bikelis [2] in general. See also [15, 19].

On the other hand, (1.2) can be sharpened and generalized by removing the hypothesis on the finiteness of the 3rd absolute moments. This may be done in terms of the truncated Lyapunov coefficients

$$R_3 = \sum_{k=1}^n \mathbb{E}\,\min\{1, |X_k|\}X_k^2.$$

While L_3 may be large and even infinite, we have $0 \le R_3 \le \min(1, L_3)$. A suitable application of Jensen's inequality leads to the lower bound $R_3 \ge \frac{c}{\sqrt{n}}$ similarly to $L_3 \ge \frac{1}{\sqrt{n}}$. An appropriate sharpening of (1.2) is

$$\Delta_n \le cR_3. \tag{1.4}$$

Representing a natural quantified form of the Lindeberg theorem, this inequality has a long and rich history. It goes back to the works by Katz [10], Petrov [17], Studnev [23, 24], Osipov [16], Feller [7], among others, although (1.4) is often stated in the equivalent setting of the normalized sums $S_n = \frac{1}{B_n} \sum_{k=1}^n \xi_k$. Let us only mention that one may take $c = 2.02$ and even $c = 1.87$, as was shown in [13], [12]; cf. also [8] for discussions and related results.

For an illustration of the advantage of (1.4) over (1.2), let us note that $R_3 \le L_{2+\delta}$ for any $\delta \in (0, 1]$, which follows from the simple inequality $\min\{1, |x|\}x^2 \le |x|^{2+\delta}$, $x \in \mathbb{R}$. Hence, (1.4) yields another useful relation

$$\Delta_n \le cL_{2+\delta},$$

which in the i.i.d. case becomes

$$\Delta_n \le c\,\frac{\beta_{2+\delta}}{n^{\delta/2}}.$$

Necessary and sufficient conditions for the validity of the rate $\Delta_n = O(n^{-\delta/2})$ in terms of the distribution of ξ_1 have been obtained by Ibragimov [9].

2 Combination of Several Lyapunov Coefficients

In general, the standard rate as in (1.3) cannot be improved, even if higher order moments of X_k are finite (for example, for normalized sums of Bernoulli random variables). Similarly, one may not replace L_3 with other Lyapunov coefficients in the more general bound (1.2). Nevertheless, in the non-i.i.d. case, (1.2) may be sharpened by using L_4 in combination with other L_p. In a typical situation, these quantities are getting smaller for growing values of p, while in general $L_p^{1/(p-2)}$ is non-decreasing in $p > 2$ (cf. Remark 6.2 below). In particular,

$$L_{2+\delta}^{1/\delta} \le L_3 \le \sqrt{L_4} \quad \text{for any } \delta \in (0, 1].$$

To describe the possible range of Δ_n, first let us complement (1.2) with two natural lower bounds for the weighted sums

$$S_n = a_1\xi_1 + \cdots + a_n\xi_n, \quad a_1^2 + \cdots + a_n^2 = 1 \quad (a_k \in \mathbb{R}), \tag{2.1}$$

of the i.i.d. random variables ξ_k with mean zero and variance one. Put $\alpha_3 = \mathbb{E}\xi_1^3$, $\beta_p = \mathbb{E}\,|\xi_1|^p$.

Theorem 2.1 *(a) Let $\alpha_3 \neq 0$ and $\beta_4 < \infty$. If the coefficients a_k in (2.1) have equal signs, then*

$$c' L_3 \leq \Delta_n \leq c L_3, \tag{2.2}$$

where the constant $c' > 0$ depends on α_3 and β_4 only.

(b) If $\beta_4 \neq 3$ and $\beta_5 < \infty$, then

$$c' L_4 \leq \Delta_n \leq c L_3, \tag{2.3}$$

where the constant $c' > 0$ depends on β_4 and β_5 only.

Thus, the Berry–Esseen bound (1.2) is sharp for the sums (2.1) with $\alpha_3 \neq 0$, when all a_k have equal signs. On the other hand, (2.2) is not applicable in the case $\alpha_3 = 0$, while (2.3) may describe a large interval which the values of Δ_n belong to. Indeed, using

$$L_p = \beta_p \left(|a_1|^p + \cdots + |a_n|^p\right) \geq \beta_p \left(\max_k |a_k|\right)^p$$

with $p = 3$, it follows that

$$L_4 \leq c L_3^{4/3}, \quad c = \beta_4 \beta_3^{-1/3}.$$

So, L_4 is essentially smaller than L_3 when the latter is small.

The main purpose of this note is to replace L_3 in (1.2) with potentially smaller quantities. Let us return to the general scheme of the sums as in (1.1).

Theorem 2.2 *Suppose that the random variables X_k have finite 4-th moments with $\mathbb{E} X_k^3 = 0$. Then, for any $\delta \in (0, 1]$,*

$$\Delta_n \leq c \left(\frac{1}{\delta} L_4 + L_{2+\delta}^{1/\delta}\right). \tag{2.4}$$

Moreover, if the distributions of X_k are symmetric about the origin and have finite absolute moments of order $2 + \delta$, then

$$\Delta_n \leq c \left(\frac{1}{\delta} R_4 + L_{2+\delta}^{1/\delta}\right). \tag{2.5}$$

Here, we use the 4-th order truncated Lyapunov coefficient

$$R_4 = \sum_{k=1}^n \mathbb{E} \min\{1, X_k^2\} X_k^2,$$

which does not require the finiteness of any absolute moments of X_k of order higher than 2 and satisfies $R_4 \leq R_3 \leq L_3$ and $R_4 \leq L_4$. Thus, the inequality (2.5) is sharper than (2.4) under the symmetry hypothesis and only requires the finiteness of absolute moments of order $2 + \delta$. Note also that (2.5) with $\delta = 1$ is equivalent to the Berry–Esseen bound (1.2).

As for the term $L_{2+\delta}^{1/\delta}$, it is not only smaller than L_3, but may also be of the same order or even smaller than R_4. On the other hand, this quantity admits a simple lower bound

$$L_{2+\delta}^{1/\delta} \geq \frac{1}{\sqrt{n}}. \tag{2.6}$$

Hence, the bounds (2.4)–(2.5) may not provide rates for Δ_n which would be better than the standard $\frac{1}{\sqrt{n}}$-rate.

Example 2.3 Let the i.i.d. random variables ξ_k have mean zero, variance one, with $\mathbb{E}\xi_1^3 = 0$ and $\beta_4 = \mathbb{E}\xi_1^4 < \infty$. We examine an asymptotic behaviour of Δ_n as $n \to \infty$ for the weighted sums

$$S_n = \frac{1}{b_n} \sum_{k=1}^n \frac{1}{k^q} \xi_k$$

with a fixed positive parameter $q < \frac{1}{2}$. The normalizing constant in front of the sum should be chosen such that

$$b_n^2 = \sum_{k=1}^n \frac{1}{k^{2q}}, \quad b_n \sim n^{\frac{1}{2}-q}.$$

Here and below, we write $Q_1 \sim Q_2$ for positive quantities $Q_j = Q_j(n)$, if $c_1 Q_1 \leq Q_2 \leq c_2 Q_1$ for all n with some constants $c_j > 0$ depending on q and β_p only.

As a main case, let $\frac{1}{3} < q < \frac{1}{2}$. Then, for any fixed $\delta \in (0, \frac{1}{q} - 2)$,

$$L_3 \sim n^{-3\,(\frac{1}{2}-q)}, \quad L_4 \sim n^{-4\,(\frac{1}{2}-q)} \sim L_3^{4/3}, \quad L_{2+\delta}^{1/\delta} \sim \frac{1}{\sqrt{n}} = o(L_3).$$

So, with this choice of δ, (2.4) is sharper than the classical Berry–Esseen bound (1.2). Moreover, as $n \to \infty$,

$$L_{2+\delta}^{1/\delta} = O(L_4) \iff q \geq \frac{3}{8}.$$

Hence, in the region $\frac{3}{8} \leq q < \frac{1}{2}$, and if $\beta_4 \neq 3$, $\beta_5 < \infty$, we get that $\Delta_n \sim L_4$, which follows from (2.4) and the lower bound in (2.3).

However, a similar conclusion cannot be made for the region $\frac{1}{4} < q < \frac{1}{3}$. Then

$$L_3 \sim L_{2+\delta}^{1/\delta} \sim \frac{1}{\sqrt{n}},$$

while $L_4 \sim n^{-4(\frac{1}{2}-q)}$ is of a smaller order.

As we will see, the inequality (2.4) may further be sharpened under higher order moment assumptions when replacing the normal distribution function $\Phi(x)$ by the corresponding Chebyshev–Edgeworth correction (this may be illustrated on the same example as above). One should emphasize, however, that this improvement may not be better than the standard rate (in view of the lower bound (2.6)). Let us mention in this connection that, for the sums S_n as in (2.1), the rate of normal approximation may be of the order $1/n$ (even in the Bernoulli case). This can be achieved either for some explicit coefficients a_k (with a certain arithmetic structure), or for typical coefficients randomly selected as coordinates of a point on the unit sphere in $\mathbb{R}^n$ (cf. [5, 11]).

In the next section we remind basic Fourier-analytic tools and discuss upper bounds for the deviations of the characteristic functions $f_n(t)$ of S_n from the standard normal characteristic function in terms of R_3 and R_4. Some technical preparations are put in Sects. 4 and 5. In Sects. 6 and 7 we collect basic properties of the truncated Lyapunov coefficients. Section 8 deals with general Gaussian-type upper bounds on $|f_n(t)|$, and then we turn to the proof of Theorem 2.2 in the symmetric case (Sect. 9). The construction of Chebyshev–Edgeworth corrections is discussed separately in Sect. 10, which are used to state and prove a more general version of the first part in Theorem 2.2 in Sect. 11. The proof of Theorem 2.1 is postponed to the last Sect. 12.

3 Berry–Esseen Bounds in Terms of Fourier–Stieltjes Transforms

The basic Fourier analytic approach to the estimation of the Kolmogorov distance

$$\rho(F, G) = \sup_x |F(x) - G(x)|$$

is a general Berry–Esseen bound

$$\rho(F, G) \leq c \int_{-T}^{T} \Big|\frac{f(t) - g(t)}{t}\Big|\, dt + \frac{cA}{T}, \tag{3.1}$$

holding true with some absolute constant $c > 0$ for all $T > 0$ (cf. e.g. [18], p. 104). Here F and G may be respectively an arbitrary non-decreasing bounded function and a function of bounded total variation on the real line with finite Lipschitz semi-norm $A = \|G\|_{\mathrm{Lip}}$ such that $F(-\infty) = G(-\infty)$, with Fourier–Stieltjes transforms

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \quad g(t) = \int_{-\infty}^{\infty} e^{itx}\, dG(x).$$

Let $S_n = X_1 + \dots + X_n$ be the sum of the independent random variables with mean zero and variances $\sigma_k^2 = \mathrm{Var}(X_k)$ such that $\sigma_1^2 + \dots + \sigma_n^2 = 1$. The relation (3.1) may be applied to the distribution function $F = F_n$ of S_n with its characteristic function

$$f_n(t) = \mathbb{E}\, e^{itS_n} = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x)$$

and with the standard normal distribution function $G = \Phi$. Then (3.1) provides a well-known upper bound for the Kolmogorov distance $\Delta_n = \rho(F_n, \Phi)$, namely

$$\Delta_n \le c \int_{-T}^{T} \Big| \frac{f_n(t) - e^{-t^2/2}}{t} \Big|\, dt + \frac{c}{T}. \tag{3.2}$$

It is also a standard fact that

$$|f_n(t) - e^{-t^2/2}| \le cL_3 \min(1, t^3)\, e^{-t^2/6}, \quad |t| \le \frac{1}{L_3}. \tag{3.3}$$

Here, the coefficient $1/6$ in the exponent may be chosen to be as close to $1/2$ as we wish by reducing the interval to the form $|t| \le \frac{c}{L_3}$ with a sufficiently small $c > 0$. Applying (3.3) in (3.2) with $T = 1/L_3$, one obtains the Berry–Esseen bound (1.2) in terms of the Lyapunov coefficient L_3.

Similarly, (1.4) follows from (3.2) with $T = 1/R_3$ and the following statement of independent interest (which is stronger and more general compared to (3.3)).

Proposition 3.1 *We have*

$$|f_n(t) - e^{-t^2/2}| \le cR_3 \min(1, t^2)\, e^{-t^2/6}, \quad |t| \le \frac{1}{32\, R_3}. \tag{3.4}$$

In a slightly different form, this relation was derived by Osipov [16] as a main step in the proof of the Berry–Esseen-type bound (1.4). In other works, (1.4) is obtained on the basis of (1.2) by using a truncation argument. Nevertheless, (3.4) is more relevant, since the finiteness of the 3rd moments of X_k is not required and since this inequality may have further applications such as local limit theorems, for example. For completeness, we will include the proof of Proposition 3.1 together with a closely related assertion in the symmetric case, which will be needed for the derivation of inequality (2.5) of Theorem 2.2.

Proposition 3.2 *Suppose that the distributions of the random variables X_k are symmetric about the origin. Then*

$$|f_n(t) - e^{-t^2/2}| \le cR_4 \min(1, t^2)\, e^{-t^2/6}, \quad |t| \le \frac{1}{32\, R_3}. \tag{3.5}$$

4 Characteristic Functions for Single Random Variables

As a next preliminary step, it is useful to fix a few elementary assertions about characteristic functions for single random variables. In this section, we assume that a random variable X has mean zero and (finite) variance $\sigma^2 = \mathrm{Var}(X)$. Introduce its characteristic function

$$f(t) = \mathbb{E}\, e^{itX}, \quad t \in \mathbb{R}.$$

Lemma 4.1 *For all $t \in \mathbb{R}$, with some complex number $\theta = \theta(t)$, $|\theta| \le 1$, we have*

$$f(t) = 1 - \frac{\sigma^2 t^2}{2} + \theta t^2\, \mathbb{E}\, \min\Big\{1, \frac{1}{2}\, |tX|\Big\}\, X^2. \tag{4.1}$$

Moreover, if the distribution of X is symmetric about the origin, then

$$f(t) = 1 - \frac{\sigma^2 t^2}{2} + \theta t^2\, \mathbb{E}\, \min\Big\{1, \frac{1}{4}\, (tX)^2\Big\}\, X^2. \tag{4.2}$$

As a consequence, we get:

Lemma 4.2 *For all $t \in \mathbb{R}$,*

$$\begin{aligned} |f(t)|^2 &\le 1 - \sigma^2 t^2 + 8t^2\, \mathbb{E}\, \min\{1, (tX)^2\}\, X^2 \\ &\le 1 - \sigma^2 t^2 + 8t^2\, \mathbb{E}\, \min\{1, |t|X|\}\, X^2. \end{aligned} \tag{4.3}$$

Proof By the moment assumptions, the characteristic function $f(t)$ has two continuous derivatives with $f'(0) = 0$ and $f''(0) = -\sigma^2$ (assuming without loss of generality that $\sigma > 0$). Hence, by the integral Taylor's formula, for all $t \in \mathbb{R}$,

$$\begin{aligned} f(t) &= 1 + t^2 \int_0^1 f''(st)\,(1-s)\, ds \\ &= 1 - \frac{\sigma^2 t^2}{2} - t^2 \int_0^1 (f''(0) - f''(st))\,(1-s)\, ds, \end{aligned}$$

which implies that

$$f(t) = 1 - \frac{\sigma^2 t^2}{2} + \frac{\theta t^2}{2} \max_{0 \le s \le |t|} |f''(0) - f''(s)| \tag{4.4}$$

with some complex number θ such that $|\theta| \le 1$.

In order to bound the last maximum, suppose that $u(t)$ is the characteristic function of a random variable with distribution function $U(x)$, that is,

$$1 - u(t) = \int_{-\infty}^{\infty} (1 - e^{itx})\, dU(x).$$

Using $|1 - e^{is}| \le 2$ and $|1 - e^{is}| \le |s|$, $s \in \mathbb{R}$, we have

$$|1 - u(t)| \le \int_{-\infty}^{\infty} \min\{2, |tx|\}\, dU(x).$$

Moreover, if the measure U is symmetric about the origin, then $u(t)$ is real-valued and the above inequality may be sharpened. In this case

$$1 - u(t) = 2\int_{-\infty}^{\infty} \sin^2\left(\frac{tx}{2}\right) dU(x)$$
$$\leq 2\int_{-\infty}^{\infty} \min\left\{1, \frac{(tx)^2}{4}\right\} dU(x).$$

Since the right-hand sides in both inequalities represent non-decreasing functions in $t \geq 0$, we respectively get stronger bounds

$$\max_{|s|\leq|t|} |1 - u(s)| \leq \int_{-\infty}^{\infty} \min\{2, |tx|\}\, dU(x),$$
$$\max_{|s|\leq|t|} (1 - u(s)) \leq 2\int_{-\infty}^{\infty} \min\left\{1, \frac{(tx)^2}{4}\right\} dU(x).$$

To obtain (4.1)–(4.2), it remains to apply these bounds in (4.4) with

$$dU(x) = \frac{1}{\sigma^2}\, x^2 dF(x), \quad u(t) = -\frac{1}{\sigma^2}\, f''(t),$$

where $F(x) = \mathbb{P}\{X \leq x\}$ denotes the distribution function of X.

Turning to the next lemma, let X' be an independent copy of X. Applying (4.2) to the random variable $Y = X - X'$, we have

$$|f(t)|^2 = 1 - \sigma^2 t^2 + \theta t^2\, \mathbb{E}\, \psi(Y), \quad \psi(x) = \min\left\{1, \frac{(tx)^2}{4}\right\} x^2. \tag{4.5}$$

The function $\psi(x)$ is non-negative, even and non-decreasing in $x > 0$, so is the function

$$w(x) = \psi(2x) = 4\min\{1, (tx)^2\}\, x^2.$$

Hence, given $x_1 \geq x_2 \geq 0$, we have

$$\psi(x_1 + x_2) \leq \psi(2x_1) = w(x_1) \leq w(x_1) + w(x_2).$$

The resulting inequality holds for $x_2 \geq x_1 \geq 0$ as well. Therefore, for all $x_1, x_2 \in \mathbb{R}$,

$$\psi(x_1 + x_2) = \psi(|x_1 + x_2|) \leq \psi(|x_1| + |x_2|) \leq w(x_1) + w(x_2).$$

Applying this subadditivity property in (4.5), we obtain that

$$\mathbb{E}\, \psi(Y) \leq \mathbb{E}\, w(X) + \mathbb{E}\, w(X') = 2\, \mathbb{E}\, w(X),$$

so that

$$|f(t)|^2 \leq 1 - \sigma^2 t^2 + 2\theta t^2 \, \mathbb{E}\, w(X).$$

□

5 Some Moment Inequalities

Towards the proof of Theorem 2.2, we will need the following moment inequality due to Cox and Kemperman [6].

Proposition 5.1 *Given independent random variables X and Y with mean zero and finite absolute moments of order $p \geq 2$, we have*

$$\mathbb{E}\,|X+Y|^p \leq 2^{p-2} \big(\mathbb{E}\,|X|^p + \mathbb{E}\,|Y|^p\big). \tag{5.1}$$

With a worse constant, (5.1) is obtained by applying Jensen's inequality. In the present formulation, it is sharp with an equality when both X and Y have a symmetric Bernoulli distribution. As was shown in [6], the inequality (5.1) follows from the "non-random" relation

$$|x+y|^p \leq 2^{p-2} \big(|x|^p + |y|^p + x\,\mathrm{sign}(y)\,|y|^{p-1} + y\,\mathrm{sign}(x)\,|x|^{p-1}\big),$$

which is valid for all $x, y \in \mathbb{R}$. For the sake of completeness, let us describe an alternative argument which covers the range $2 \leq p \leq 4$. It is based on the following:

Lemma 5.2 *Let $2 \leq p \leq 4$. If X' and Y' are respectively independent copies of independent random variables X and Y with mean zero, then*

$$\mathbb{E}\,|X+Y|^p \leq \frac{1}{2}\,\mathbb{E}\,|X-X'|^p + \frac{1}{2}\,\mathbb{E}\,|Y-Y'|^p. \tag{5.2}$$

This interesting relation was obtained by Ushakov in [27], where it was additionally assumed that X and Y have symmetric distributions, and by Pinelis [20] in the general case. Their proofs are similar and short, so, we reproduce here.

Proof Given a random variable X with finite $\beta_p = \mathbb{E}\,|X|^p$, $p > 0$, define the moments $\alpha_k = \mathbb{E}X^k$ for integers $0 \leq k \leq p$ (with the convention that $\alpha_0 = 1$). It is known that the moment β_p may be expressed in terms of the characteristic function $f(t) = \mathbb{E}\,e^{itX}$. The following representation was given by von Bahr [28]: If p is not an even integer, then

$$\beta_p = C(p) \int_{-\infty}^{\infty} \bigg[\mathrm{Re}(f(t)) - \sum_{k=0}^{[p/2]} (-1)^k \alpha_{2k}\, \frac{t^{2k}}{(2k)!}\bigg] \frac{dt}{t^{p+1}}, \tag{5.3}$$

where

$$C(p) = \frac{1}{\pi}\,\Gamma(p+1)\,\cos\Big(\frac{(p+1)\pi}{2}\Big). \tag{5.4}$$

In particular, $C(p) > 0$ for $2 < p < 4$, and if X has mean zero and variance $\sigma^2 = \mathrm{Var}(X)$, the equality (5.3) takes the form

$$\beta_p = C(p) \int_{-\infty}^{\infty} \Big[\mathrm{Re}(f(t)) - 1 + \frac{\sigma^2 t^2}{2}\Big]\,\frac{dt}{t^{p+1}}. \tag{5.5}$$

Moreover, it was shown by Ushakov [26], p. 89, that in the case where X has mean zero and variance σ^2, we have

$$\mathrm{Re}(f(t)) \geq 1 - \frac{\sigma^2 t^2}{2}$$

for all $t \in \mathbb{R}$. Hence, the integrand in (5.5) is non-negative.

Returning to (5.2), we may assume that X and Y have finite absolute moments of order $p \in (2,4)$. Put $\sigma_1^2 = \mathrm{Var}(X)$, $\sigma_2^2 = \mathrm{Var}(Y)$, so that $X+Y$ has variance $\sigma^2 = \sigma_1^2 + \sigma_2^2$. Let $f_1(t)$ and $f_2(t)$ be the characteristic functions of X and Y, respectively. Then $X - X'$ and $Y - Y'$ have characteristic functions $|f_1(t)|^2$ and $|f_2(t)|^2$, while $X+Y$ has characteristic function $f_1(t) f_2(t)$. Using

$$|f_1(t)|^2 + |f_2(t)|^2 \;\geq\; 2\,|f_1(t) f_2(t)| \;\geq\; 2\,\mathrm{Re}(f_1(t) f_2(t))$$

and applying (5.5) to $X - X'$, $Y - Y'$ and $X+Y$, it follows that

$$\begin{aligned}
\mathbb{E}\,|X - X'|^p + \mathbb{E}\,|Y - Y'|^p &= C(p) \int_{-\infty}^{\infty} \Big[|f_1(t)|^2 - 1 + \sigma_1^2 t^2\Big]\,\frac{dt}{t^{p+1}} \\
&\quad + C(p) \int_{-\infty}^{\infty} \Big[|f_2(t)|^2 - 1 + \sigma_2^2 t^2\Big]\,\frac{dt}{t^{p+1}} \\
&\geq\; 2C(p) \int_{-\infty}^{\infty} \Big[\mathrm{Re}(f_1(t) f_2(t)) - 1 + \frac{\sigma^2 t^2}{2}\Big]\,\frac{dt}{t^{p+1}} \;=\; 2\,\mathbb{E}\,|X+Y|^p.
\end{aligned}$$

□

Proof of Proposition 5.1 ($2 \leq p \leq 4$). As a first step, let us show that, if a random variable X takes at most two values, and X' is an independent copy of X, then, for any $p \geq 2$,

$$\mathbb{E}\,|X - X'|^p \leq 2^{p-1}\,\mathbb{E}\,|X|^p. \tag{5.6}$$

Note that, by Jensen's inequality, one has a similar relation with an additional factor of 2.

Suppose that X takes two non-zero values x_1 and x_2 with respective probabilities $q_1 > 0$ and $q_2 > 0$. Then the inequality of the form $\mathbb{E}\,|X - X'|^p \le c\,\mathbb{E}\,|X|^p$ may be rewritten as

$$2q_1q_2\,|x_1 - x_2|^p \le c\,(q_1|x_1|^p + q_2|x_2|^p).$$

Here the worst case is attained for

$$q_1 = \frac{|x_2|^{p/2}}{|x_1|^{p/2} + |x_2|^{p/2}}, \quad q_2 = \frac{|x_1|^{p/2}}{|x_1|^{p/2} + |x_2|^{p/2}},$$

and then we are reduced to

$$2\,|x_1 - x_2|^p \le c\left(|x_1|^{p/2} + |x_2|^{p/2}\right)^2.$$

It is easy to see that this inequality holds true with best constant $c = 2^{p-1}$.

Turning to the inequality (5.1), we assume that $2 < p < 4$ and that both X and Y are bounded and take values in some closed interval Δ. Denote by μ and ν the distributions of X and Y and rewrite (5.1) as

$$2^{2-p}\int_\Delta |x + y|^p\,d\mu(x)\,d\nu(y) \le \int_\Delta |x|^p\,d\mu(x) + \int_\Delta |x|^p\,d\nu(x).$$

Since it is bi-linear with respect to (μ, ν), it is sufficient to verify this inequality for all extreme points in the space of all probability measures on Δ with barycenter at the origin. But, such points have at most two atoms, in view of the linear constraint $\int_\Delta x\,d\mu(x) = 0$. As a consequence, we are reduced in (5.1) to the case where the random variables X and Y take at most two values. In this case, let X' and Y' be independent copies of X and Y, respectively. Combining the inequalities (5.2) and (5.6), we then get

$$\mathbb{E}\,|X + Y|^p \;\le\; \frac{1}{2}\,\mathbb{E}\,|X - X'|^p + \frac{1}{2}\,\mathbb{E}\,|Y - Y'|^p \;\le\; 2^{p-2}\,\mathbb{E}\,|X|^p + 2^{p-2}\,\mathbb{E}\,|Y|^p.$$

□

6 Truncated Lyapunov Coefficients

Let us now return to the scheme of independent random variables $X_1, \dots, X_n$ that are defined on some probability space $(\Omega, \mathbb{P})$ and are such that $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 = \sigma_k^2$, $\sigma_1^2 + \cdots + \sigma_n^2 = 1$.

The truncated Lyapunov coefficient of order $p > 2$ for the sequence $(X_k)_{k \le n}$ is defined by

$$R_p = \sum_{k=1}^n \mathbb{E}\,\min\{1, |X_k|^{p-2}\}\,X_k^2.$$

More generally, define the truncated Lyapunov function by

$$R_p(t) = \sum_{k=1}^{n} \mathbb{E}\, \min\{1, |tX_k|^{p-2}\}\, X_k^2, \quad t \in \mathbb{R}, \tag{6.1}$$

so that $R_p = R_p(1)$. Note that $0 \le R_p(t) \le 1$, $R_p(0) = 0$, $R_p(\infty) = 1$. Hence $R_p(t)$ may be treated as a distribution function.

In the special case $p = 3$, it is connected with the Lindeberg function

$$L(x) = \sum_{k=1}^{n} \int_{|y| \ge x} y^2\, dF_k(y),$$

where $F_k(x) = \mathbb{P}\{X_k \le x\}$ stand for the distribution functions of X_k. Namely,

$$R_3(t) = |t| \int_0^{\frac{1}{|t|}} L(x)\, dx.$$

In addition, $\lim_{p \to \infty} R_p(t) = L(1/|t|)$.

Let us give a few basic properties of the truncated Lyapunov functions.

Proposition 6.1 *For each $t \in \mathbb{R}$, the function $p \to R_p(t)$ is non-increasing, while the function $p \to R_p(t)^{\frac{1}{p-2}}$ is non-decreasing in $p > 2$. In particular,*

$$R_4(t) \le R_3(t) \le R_4(t)^{1/2}. \tag{6.2}$$

Proof The first claim is obvious. Let ξ be a random variable with distribution

$$dF(x) = \sum_{k=1}^{n} x^2\, dF_k(x).$$

Then

$$R_p(t)^{\frac{1}{p-2}} = \Big(\mathbb{E}\, \min(1, |t\xi|)^{p-2}\Big)^{\frac{1}{p-2}}.$$

Here, the right-hand side represents the L^{p-2}–norm of the random variable $\min(1, |t\xi|)$, so, it is non-decreasing in p. □

Remark 6.2 The second claim in Proposition 6.1 is analogous to the property that the function $p \to L_p^{\frac{1}{p-2}}$ is non-decreasing in $p > 2$. This follows from the representation

$$L_p^{\frac{1}{p-2}} = \big(\mathbb{E}\, |\xi|^{p-2}\big)^{\frac{1}{p-2}}.$$

Proposition 6.3 *There is a smallest value $T \in (0, \infty]$ such that $R_p(t)$ is increasing and continuous in $0 \le t < T$, with $R_p(T-) = 1$. Moreover, T does not depend on p.*

Proof Clearly, $R_p(t)$ is non-decreasing and continuous in $t \ge 0$, by the Lebesgue dominated convergence theorem. Moreover, suppose that it is constant on some interval, that is,

$$\sum_{k=1}^{n} \mathbb{E} \min\{1, |tX_k|^{p-2}\} X_k^2 = \sum_{k=1}^{n} \mathbb{E} \min\{1, |sX_k|^{p-2}\} X_k^2$$

for some $0 < t < s$. Then a.s.

$$\sum_{k=1}^{n} \min\{1, |tX_k|^{p-2}\} X_k^2 = \sum_{k=1}^{n} \min\{1, |sX_k|^{p-2}\} X_k^2.$$

But this is equivalent to the statement that a.s. $\min\{1, t|X_k|\} = \min\{1, s|X_k|\}$ for any $k \le n$. If $X_k(\omega) \neq 0$ for some $\omega \in \Omega$, the latter is only possible when $t|X_k(\omega)| \ge 1$, that is,

$$t \ge T \equiv \max_{1 \le k \le n} \operatorname*{ess\,sup}_{\omega \in \Omega} \left[\frac{1}{|X_k(\omega)|} 1_{\{X_k(\omega) \neq 0\}} \right].$$

In addition, if T is finite and $t \ge T$, then $R_p(t) = 1$. □

Proposition 6.4 *For any $p > 2$,*

$$R_p(t) \le 2|t|^{p-2} R_p, \quad |t| \ge 1. \tag{6.3}$$

Proof Recalling the definition (6.1), introduce

$$u(t) = R_p(t^{\frac{1}{p-2}}) = \sum_{k=1}^{n} \mathbb{E} \min\{1, t|X_k|^{p-2}\} X_k^2, \quad t \ge 0.$$

It is a continuous, non-decreasing function such that $u(0) = 0$. Moreover, it is concave due to the concavity of the functions $t \to \min\{1, t|X_k|^{p-2}\}$. Therefore, u is subadditive:

$$u(s_1 + \cdots + s_l) \le u(s_1) + \cdots + u(s_l) \quad \text{for all } s_1, \dots, s_l \ge 0.$$

In particular, $u(ls) \le lu(s)$ for all $s \ge 0$. Hence, for all $t \ge 1$, putting $l = 2\,[t]$ and $s = t/l$, we have $u(t) = u(ls) \le 2\,[t]\,u(1) \le 2t\,u(1)$, and (6.3) follows. □

Proposition 6.5 *For any $p > 2$ and $\alpha \in [0, 1)$, the equation $R_p(t) = \alpha$ has a unique solution $t \in [0, \infty)$. Moreover, if $R_p \le \alpha$, then*

$$t \ge \left(\frac{\alpha}{2R_p}\right)^{\frac{1}{p-2}}. \tag{6.4}$$

Proof By Proposition 6.3, the inequality $R_p(t) \le \alpha$ is equivalent to $t \in [-T, T]$ for a certain number $T > 0$ such that $R_p(T) = \alpha$ and $R_p(t) > \alpha$ for $|t| > T$. Here necessarily $T \ge 1$ in the case $R_p \le \alpha$. Then, applying (6.3) with $t = T$, we get $\alpha = R_p(T) \le 2T^{p-2} R_p$, which is the same as (6.4). □

7 Bounds on Variances in Terms of Lyapunov Coefficients

Let us keep notations and assumptions as in the previous section. The Lyapunov coefficients

$$L_p = \sum_{k=1}^{n} \mathbb{E}\, |X_k|^p, \quad p > 2, \tag{7.1}$$

may be used to control the variances $\sigma_k^2 = \mathbb{E} X_k^2$. Indeed, since $\mathbb{E}\, |X_k|^p \ge (\mathbb{E}\, X_k^2)^{p/2}$, it follows from (7.1) that

$$\max_{1 \le k \le n} \sigma_k \le \Big(\sum_{k=1} \sigma_k^p\Big)^{1/p} \le L_p^{1/p}.$$

Thus, the smallness of L_p implies that all variances σ_k^2 are uniformly small.

We need a certain analog of this property for the truncated Lyapunov coefficients, as well as for the functions

$$R_p(t) = \sum_{k=1}^{n} \mathbb{E}\, \min\{1, |tX_k|^{p-2}\}\, X_k^2, \quad t \in \mathbb{R}.$$

Given $p > 2$, $t \neq 0$, define $q = \frac{p-2}{2}$, $s = |t|^{p-2}$, and consider

$$u(y) = \min\{1, sy^q\}\, y, \quad y \ge 0.$$

This function is nearly convex and therefore satisfies a weak form of Jensen's inequality. Indeed, it has derivative

$$u'(y) = \begin{cases} s(q+1)\, y^q & \text{for } 0 < y < s^{-1/q}, \\ 1 & \text{for } y > s^{-1/q}, \end{cases}$$

with $u'(s^{-1/q}-) = \frac{p}{2} > 1$, which shows that u is not convex. Let us modify it to get a convex function. Put

$$y_0 = (s(q+1))^{-1/q}$$

and define the function $\tilde{u}$ on the positive half-axis by requiring that $\tilde{u}(0) = 0$ and

$$\tilde{u}'(y) = \begin{cases} s(q+1)\, y^q & \text{for } 0 < y < y_0, \\ 1 & \text{for } y > y_0. \end{cases}$$

By the construction,

$$\tilde{u}(y) = \begin{cases} s\, y^{q+1} & \text{for } 0 \le y \le y_0, \\ s\, y_0^{q+1} + (y - y_0) & \text{for } y \ge y_0. \end{cases}$$

In particular, $\tilde{u}(y) = u(y)$ for $0 \le y \le y_0$, while on the interval $y_0 \le y \le s^{-1/q}$,

$$\frac{\tilde{u}(y)}{u(y)} = \frac{y - \frac{q}{q+1}\, y_0}{s\, y^{q+1}} = \frac{1}{s}\, y^{-q} - \frac{q y_0}{s(q+1)}\, y^{-q-1} \equiv g(y).$$

We have

$$g'(y) = -\frac{q}{s}\, y^{-q-1} + \frac{q y_0}{s}\, y^{-q-2} = 0 \iff y = y_0.$$

This shows that $g(y)$ is monotone on this interval with values at the end points

$$g(y_0) = 1, \quad g(s^{-1/q}) = 1 - \frac{q}{(q+1)^{1+1/q}} \equiv d(q) < 1.$$

Also, on the interval $y \ge s^{-1/q}$,

$$\frac{\tilde{u}(y)}{u(y)} = \frac{s\, y_0^{q+1} + (y - y_0)}{y} = 1 - \frac{q y_0}{q+1}\, y^{-1}$$

which is an increasing function. This implies that

$$\tilde{u}(y) \ge d(q)\, u(y) \quad \text{for all } \ y \ge 0,$$

with equality attainable at $y = s^{-1/q}$.

One may now apply Jensen's inequality. Since $u \ge \tilde{u}$, while $\tilde{u}$ is convex, we get

$$\begin{aligned} \mathbb{E}\, \min\{1, |tX_k|^{p-2}\} X_k^2 &= \mathbb{E}\, u(X_k^2) \ \ge\ \mathbb{E}\, \tilde{u}(X_k^2) \\ &\ge \tilde{u}(\sigma_k^2) \ \ge\ d(q)\, u(\sigma_k^2) \ =\ d(q)\, \min\{1, |t\sigma_k|^{p-2}\}\, \sigma_k^2. \end{aligned}$$

One may summarize. Note that $1 - d(q) = \frac{p-2}{p}\, (\frac{2}{p})^{\frac{2}{p-2}}$.

Lemma 7.1 *For every $t \in \mathbb{R}$ and $k \le n$,*

$$c_p\, \min\{1, (|t|\sigma_k)^{p-2}\}\, \sigma_k^2 \ \le\ \mathbb{E}\, \min\{1, |tX_k|^{p-2}\}\, X_k^2$$

with constant

$$c_p = 1 - \frac{p-2}{p}\left(\frac{2}{p}\right)^{\frac{2}{p-2}}.$$

In particular,

$$\sum_{k=1}^{n} \sigma_k^3 \leq \frac{27}{23}\, R_3, \quad \sum_{k=1}^{n} \sigma_k^4 \leq \frac{4}{3}\, R_4.$$

More generally,

$$R_p \geq c_p \sum_{k=1}^{n} \sigma_k^p \geq c_p\, n^{-\frac{p-2}{2}},$$

where the equality in the last inequality is attained for equal variances $\sigma_k^2 = 1/n$.

8 Upper Bounds for the Product of Characteristic Functions

As before, let $X_1, \dots, X_n$ be independent random variables with mean zero and variances $\sigma_k^2 = \mathbb{E}X_k^2$ such that $\sigma_1^2 + \dots + \sigma_n^2 = 1$. Then the sum $S_n = X_1 + \dots + X_n$ has mean zero, variance one, and characteristic function

$$f_n(t) = v_1(t) \dots v_n(t), \quad t \in \mathbb{R},$$

where $v_k(t) = \mathbb{E}\, e^{itX_k}$ denote the characteristic functions of X_k.

Lemma 4.2 and Propositions 6.3–6.5 can be used to bound the absolute value of $f_n(t)$.

Proposition 8.1 *We have*

$$|f_n(t)| \leq 2e^{-t^2/4}, \quad |t| \leq \frac{1}{32R_3}. \tag{8.1}$$

Proof By the inequality (4.3) applied to X_k, and using $1 + x \leq e^x$ ($x \in \mathbb{R}$), we have

$$\begin{aligned} |v_k(t)|^2 &\leq 1 - \sigma_k^2 t^2 + 8t^2\, \mathbb{E}\, \min\{1, |tX_k|\}\, X_k^2 \\ &\leq \exp\{-\sigma_k^2 t^2 + 8t^2\, \mathbb{E}\, \min\{1, |tX_k|\}\}\, X_k^2. \end{aligned}$$

Multiplying these inequalities, we obtain that

$$|f_n(t)| \leq \exp\Big\{-\frac{t^2}{2} + 4t^2 R_3(t)\Big\}.$$

Let T be the positive solution to the equation $R_3(T) = \frac{1}{16}$. Hence, in the interval $|t| \le T$, there is a subgaussian bound

$$|f_n(t)| \le e^{-t^2/4}. \tag{8.2}$$

Note that $T \ge \frac{1}{32\,R_3}$ as long as $R_3 \le \frac{1}{16}$, according to Proposition 6.5 with $p = 3$ and $\alpha = \frac{1}{16}$. Thus, (8.1) is fulfilled in the interval $|t| \le \frac{1}{32\,R_3}$ if $R_3 \le \frac{1}{16}$.

In the case $R_3 > \frac{1}{16}$, the inequality (8.2) remains valid in the same interval in a slightly weaker form such as (8.1). Then $|t| \le \frac{1}{32\,R_3} < \frac{1}{2}$ and therefore the right-hand side of (8.1) is greater than $2 \cdot e^{-1/16} > 1$. □

It is well-known that the inequality (8.1) holds true for $|t| \le c/L_3$. In fact, this interval may be enlarged in terms of other Lyapunov coefficients, if we allow a slower decay.

Proposition 8.2 *For all* $\delta \in (0, 2]$, *we have*

$$|f_n(t)| \le e^{-\delta t^2/6}, \quad |t| \le \frac{1}{L_{2+\delta}^{1/\delta}}. \tag{8.3}$$

In particular,

$$|f_n(t)| \le e^{-t^2/6}, \quad |t| \le \frac{1}{L_3}.$$

Proof Suppose that every summand X_k has a finite absolute moment of order $2 + \delta$. We employ Proposition 5.1 which provides the moment inequality

$$\mathbb{E}\,|X_k - Y_k|^{2+\delta} \le 2^{1+\delta}\,\mathbb{E}\,|X_k|^{2+\delta}, \quad 0 \le \delta \le 2, \tag{8.4}$$

where Y_k is an independent copy of X_k.

We need an upper bound for the cosine function of the form

$$\cos x \le 1 - \frac{1}{2}\,x^2 + \frac{1}{2}\,c_\delta\,|x|^{2+\delta}, \quad x \in \mathbb{R}. \tag{8.5}$$

As was shown by Ushakov for the range $0 < \delta \le 1$ (cf. [26], Lemma 2.1.10), this holds with $c_\delta = \frac{2}{(2+\delta)\,\delta^{1+\delta}}\,(\theta - \sin\theta)$, where $\theta \in (0, 2\pi)$ is the unique solution to the equation

$$\frac{\delta}{2(2+\delta)}\,\theta^2 + \frac{1}{2+\delta}\,\theta\sin\theta + \cos\theta = 1.$$

Let us derive a simple explicit expression for (non-optimal) constants in (8.5) which will be sufficient for our purposes. By Taylor's formula, for all $x \in \mathbb{R}$,

$$\cos x \le 1 - \frac{1}{2}\,x^2 + \frac{1}{24}\,x^4.$$

Fix a parameter $a > 1$. For all $|x| \le a$,

$$\frac{1}{24}\,x^4 \le \frac{1}{2}\,c\,|x|^{2+\delta} \iff \frac{1}{12}\,a^{2-\delta} \le c.$$

Hence, in this interval one may put $c_\delta = \frac{a^2}{12} \cdot a^{-\delta}$. For $|x| \ge a$, one may use $\cos x \le 1$ and

$$1 \le 1 - \frac{1}{2}\,x^2 + \frac{1}{2}\,c\,|x|^{2+\delta} \iff |x|^{-\delta} \le c.$$

Hence, in this region one may put $c_\delta = a^{-\delta}$. Equalizing the two choices, one may take $a = \sqrt{12}$. Thus, we obtain (8.5) in the form

$$\cos x \le 1 - \frac{1}{2}\,x^2 + \frac{1}{2} \cdot 12^{-\delta/2}\,|x|^{2+\delta}.$$

As a consequence, applying this inequality with $x = t(X_k - Y_k)$ and then (8.4), we get

$$\begin{aligned} |v_k(t)|^2 &= \mathbb{E}\,\cos(t(X_k - Y_k)) \\ &\le 1 - \sigma_k^2 t^2 + \frac{1}{2} \cdot 12^{-\delta/2}\,|t|^{2+\delta}\,\mathbb{E}\,|X_k - Y_k|^{2+\delta} \\ &\le 1 - \sigma_k^2 t^2 + 3^{-\delta/2}\,|t|^{2+\delta}\,\mathbb{E}\,|X_k|^{2+\delta} \\ &\le \exp\Big\{ - \sigma_k^2 t^2 + 3^{-\delta/2}\,|t|^{2+\delta}\,\mathbb{E}\,|X_k|^{2+\delta}\Big\}. \end{aligned}$$

Thus, for all $t \in \mathbb{R}$,

$$|v_k(t)| \le \exp\Big\{ - \frac{\sigma_k^2 t^2}{2} + \frac{1}{2} \cdot 3^{-\delta/2}\,|t|^{2+\delta}\,\mathbb{E}\,|X_k|^{2+\delta}\Big\}.$$

Multiplying these inequalities over $k = 1, \dots, n$, we conclude that

$$|f_n(t)| \le \exp\Big\{ - \frac{1}{2}\,t^2 + \frac{1}{2} \cdot 3^{-\delta/2}\,|t|^{2+\delta}\,L_{2+\delta}\Big\}.$$

As a result, if $|t|^\delta\,L_{2+\delta} \le 1$, we arrive at the general subgaussian bound

$$|f(t)| \le \exp\Big\{ - \frac{1}{2}\,(1 - 3^{-\delta/2})\,t^2\Big\}.$$

To simplify, one may use $1 - 3^{-\delta/2} \ge \frac{1}{3}\,\delta$ for the range $0 < \delta \le 2$, which leads to (8.3). □

9 Proof of Propositions 3.1 and 3.2 and Theorem 2.2 (Symmetric Case)

Our next step is to derive an approximation for the product characteristic function

$$f_n(t) = v_1(t) \dots v_n(t)$$

by the characteristic function of the standard normal law, in which the error terms are estimated by means of the truncated Lyapunov coefficients R_3 and R_4. First, we establish the bounds in Propositions 3.1–3.2 on smaller intervals.

As before, we denote by $v_k(t) = \mathbb{E}\, e^{itX_k}$ the characteristic functions of the independent random variables X_k with mean zero and variances σ_k^2 such that $\sigma_1^2 + \dots + \sigma_n^2 = 1$.

Lemma 9.1 *We have*

$$|f_n(t) - e^{-t^2/2}| \le cR_3 \max(t^2, |t|^3)\, e^{-t^2/2}, \qquad |t| \le R_3^{-1/3}. \tag{9.1}$$

Moreover, if the distributions of all X_k are symmetric about the origin, then

$$|f_n(t) - e^{-t^2/2}| \le cR_4 \max(t^2, t^4)\, e^{-t^2/2}, \qquad |t| \le R_4^{-1/4}. \tag{9.2}$$

We employ the following elementary assertion.

Lemma 9.2 *Given complex numbers z_k, $1 \le k \le n$, we have*

$$\Big|\prod_{k=1}^n (1+z_k) - 1\Big| \le e^a - 1, \quad a = \sum_{k=1}^n |z_k|.$$

Proof Write

$$\prod_{k=1}^n (1+z_k) - 1 \;=\; \sum_{k=1}^n \; \sum_{1 \le i_1 < \dots < i_k \le n} z_{i_1} \dots z_{i_k}.$$

For every $k \le n$, the inner sum does not exceed in absolute value the number

$$\frac{1}{k!} \sum_{i_1 \ne \dots \ne i_k} |z_{i_1}| \dots |z_{i_k}| \;\le\; \frac{1}{k!} \sum_{1 \le i_1, \dots, i_k \le n} |z_{i_1}| \dots |z_{i_k}| \;=\; \frac{a^k}{k!}.$$

Hence

$$\Big|\prod_{k=1}^n (1+z_k) - 1\Big| \;\le\; \sum_{k=1}^n \frac{a^k}{k!} \;\le\; e^a - 1.$$

□

Proof of Lemma 9.1 We use Lemma 9.2 to quantify the closeness of the product

$$e^{t^2/2} f_n(t) = \prod_{k=1}^{n} e^{\sigma_k^2 t^2/2} v_k(t)$$

to 1 on corresponding t-intervals. By Lemma 7.1,

$$\max_{1\le k\le n} (\sigma_k |t|)^3 \le |t|^3 \sum_{k=1}^{n} \sigma_k^3 \le \frac{27}{23}\, |t|^3 R_3 \le \frac{27}{23}$$

for $|t| \le R_3^{-1/3}$ and

$$\max_{1\le k\le n} (\sigma_k t)^4 \le t^4 \sum_{k=1}^{n} \sigma_k^4 \le \frac{4}{3}\, t^4 R_4 \le \frac{4}{3} \tag{9.3}$$

for $|t| \le R_4^{-1/4}$ in the second scenario. In both cases,

$$\sigma_k |t| \le \alpha = \Big(\frac{4}{3}\Big)^{1/4} < 1.1, \quad 1 \le k \le n,$$

which implies that $e^{\sigma_k^2 t^2/2} \le e^{\alpha^2/2} < 2$.

Now, applying the representations (4.1)–(4.2) of Lemma 4.1 to the random variable X_k, we obtain that, for some $\theta_k = \theta_k(t)$, $|\theta_k| \le 1$,

$$z_k(t) \equiv e^{\sigma_k^2 t^2/2} v_k(t) = e^{\sigma_k^2 t^2/2}\Big(1 - \frac{\sigma_k^2 t^2}{2}\Big) + e^{\sigma_k^2 t^2/2}\, \delta_k(t) \tag{9.4}$$

with

$$\delta_k(t) = \theta_k t^2\, \mathbb{E}\, \min\{1, |t X_k|\}\, X_k^2$$

in general, and with

$$\delta_k(t) = \theta_k t^2\, \mathbb{E}\, \min\{1, (t X_k)^2\}\, X_k^2$$

in the symmetric case.

The function $w(s) = e^s(1-s)$ appearing on the right-hand side of (9.4) satisfies $w(0) = 1$, $w'(0) = 0$, $w''(s) = -e^s(1+s)$. Hence, by Taylor's formula,

$$|w(s) - 1| \le \frac{1}{2}\, e^{s_0}(1+s_0)\, s^2, \quad 0 \le s \le s_0.$$

Applying this inequality with $s = \sigma_k^2 t^2/2$ and $s_0 = \alpha^2/2$, we get

$$\Big|e^{\sigma_k^2 t^2/2}\Big(1 - \frac{\sigma_k^2 t^2}{2}\Big) - 1\Big| \le \frac{1}{8}\, e^{\alpha^2/2}\Big(1 + \frac{\alpha^2}{2}\Big)\, \sigma_k^4 t^4 \le \frac{1}{2}\, \sigma_k^4 t^4,$$

and (9.4) gives

$$|z_k(t) - 1| \le \frac{1}{2}\, \sigma_k^4 t^4 + 2\, |\delta_k(t)|.$$

One may now apply Lemma 9.2 with $z_k = z_k(t) - 1$, which yields

$$|z - 1| \le e^a - 1, \qquad z = f_n(t)\, e^{t^2/2}, \quad a = \sum_{k=1}^n \Big(\frac{1}{2}\, \sigma_k^4 t^4 + 2\, |\delta_k(t)|\Big). \tag{9.5}$$

Recall that, by Lemma 7.1 with $p = 4$ and the constant $c_4 = 3/4$,

$$R_4(t) \ \ge\ c_4 \sum_{k=1}^n \min\{1, (t\sigma_k)^2\}\, \sigma_k^2 \ \ge\ \frac{c_4}{1.1^2} \sum_{k=1}^n t^2 \sigma_k^4 \ \ge\ \frac{1}{2}\, t^2 \sum_{k=1}^n \sigma_k^4.$$

Hence, for the first claim of Lemma 9.1, from (9.5) we have

$$a \le t^2 R_4(t) + 2t^2 R_3(t) \le 3t^2 R_3(t), \tag{9.6}$$

where we used $R_4(t) \le R_3(t)$. In the case $|t| \ge 1$, we employ the relation $R_3(t) \le 2|t|R_3$ from Proposition 6.4 with $p = 3$. This gives $a \le 6\, |t|^3 R_3 \le 6$. Since

$$e^a - 1 \le \frac{e^6 - 1}{6}\, a, \quad 0 \le a \le 6,$$

we get, applying (9.5) and the previous bound $a \le 6\, |t|^3 R_3$,

$$|z - 1| \le \frac{e^6 - 1}{6} \cdot 6\, |t|^3 R_3 = (e^6 - 1)\, |t|^3 R_3.$$

This is the required relation (9.1). If $|t| \le 1$, by (9.6), we have $a \le 3R_3(t) \le 3$. Since

$$e^a - 1 \le \frac{e^3 - 1}{3}\, a, \quad 0 \le a \le 3,$$

we get, applying (9.5) and the bounds $a \le 3t^2 R_3(t) \le 3t^2 R_3$,

$$|z - 1| \le \frac{e^3 - 1}{3} \cdot 3t^2 R_3 = (e^3 - 1)\, t^2 R_3.$$

Thus, (9.1) is proved for all t in the region $|t| \le R_3^{-1/3}$.

Returning to (9.5), for the second claim of the lemma we have

$$a \le 3t^2 R_4(t), \tag{9.7}$$

where we applied Lemma 7.1 once more. If $|t| \ge 1$, one may use the relation $R_4(t) \le 2t^2 R_4$ from (6.3) with $p = 4$. This gives $a \le 6\,t^4 R_4 \le 6$. Thus, by (9.5) and using a similar argument as in the previous step, we get

$$|z - 1| \le (e^6 - 1)\, t^4 R_4.$$

If $|t| \le 1$, (9.7) yields $a \le 3$. Using $R_4(t) \le R_4$, we also have $a \le 3t^2 R_4$. By (9.5), both estimates imply $|z-1| \le (e^3 - 1)\, t^2 R_4$. Thus, (9.2) is proved for all $|t| \le R_4^{-1/4}$. □

Proof of Propositions 3.1–3.2 The inequality (9.1) implies a bound of the form

$$|f_n(t) - e^{-t^2/2}| \le c R_3 \min(1, t^2)\, e^{-t^2/6} \tag{9.8}$$

in the interval $|t| \le R_3^{-1/3}$, while, by Proposition 8.1,

$$|f_n(t)| \le 2e^{-t^2/4}, \quad |t| \le \frac{1}{32\, R_3}. \tag{9.9}$$

Hence, in order to extend (9.8) to the interval as in (9.9), we only need a bound

$$2e^{-t^2/4} + e^{-t^2/2} \le c R_3\, e^{-t^2/6}$$

for the region $|t| \ge R_3^{-1/3}$. Since $R_3 \le 1$, the latter is obvious.

Similarly, (9.2) implies an upper bound of the form

$$|f_n(t) - e^{-t^2/2}| \le c R_4 \min(1, t^2)\, e^{-t^2/6} \tag{9.10}$$

for $|t| \le R_4^{-1/4}$. In view of (9.9), in order to extend the latter bound to the interval as in (9.9), we only need a relation

$$2e^{-t^2/4} + e^{-t^2/2} \le c R_4\, e^{-t^2/6}$$

for the region $|t| \ge R_4^{-1/4}$. Since $R_4 \le 1$, the latter is clear as well. □

Proof of Theorem 2.2 (the symmetric case). We are prepared to derive the inequality (2.5). Put $T_0 = 1/(32\, R_3)$ and choose $T = L_{2+\delta}^{-1/\delta}$ in the Berry–Esseen inequality (3.2) with a fixed value $\delta \in (0, 1]$. Then we get

$$c\Delta_n \le \int_{-T_0}^{T_0} \Big|\frac{f_n(t) - e^{-t^2/2}}{t}\Big|\, dt + \int_{T_0 < |t| < T} \Big|\frac{f_n(t) - e^{-t^2/2}}{t}\Big|\, dt + \frac{1}{T}. \tag{9.11}$$

Here, the first integral does not exceed a multiple of R_4, according to Proposition 3.2, cf. (9.10). Applying the inequality (8.3) of Proposition 8.2, we also see that the second integral does not exceed

$$2\int_{T_0}^{\infty} \frac{e^{-\delta t^2/6}}{t}\, dt \leq c\, e^{-\delta T_0^2/6} \leq \frac{c'}{\delta T_0^2}.$$

It remains to recall that $R_3^2 \leq R_4$, cf. (6.2). □

10 Chebyshev–Edgeworth Corrections

If the random variables X_k have finite absolute moments of an integer order $p \geq 4$, the approximation for the characteristic functions $f_n(t)$ as in (3.4) may be sharpened on the interval $|t| \leq 1/L_3$ by means of the Lyapunov coefficient L_p. However, for this aim one should properly modify the standard normal characteristic function $g(t) = e^{-t^2/2}$. Namely, put

$$g_{p-1}(t) = e^{-t^2/2} + e^{-t^2/2} \sum \frac{1}{k_1! \dots k_{p-3}!} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_{p-1}}{(p-1)!}\Big)^{k_{p-3}} (it)^k \tag{10.1}$$

with

$$k = 3k_1 + \cdots + (p-1)k_{p-3},$$

where the summation runs over all collections of non-negative integers $k_1, \dots, k_{p-3}$ that are not all zero and are such that

$$k_1 + 2k_2 + \cdots + (p-3)k_{p-3} \leq p-3.$$

The definition (10.1) involves the cumulants

$$\gamma_r = \gamma_r(S_n) = \sum_{k=1}^n \gamma_r(X_k), \quad \gamma_r(X_k) = \frac{d^r}{i^r\, dt^r} \log \mathbb{E}\, e^{itX_k}\big|_{t=0},$$

which are well-defined for $r = 1, 2, \dots, p$. Every cumulant $\gamma_r(X_k)$ may be represented as a polynomial in the first r moments of X_k. Note, however, that only the cumulants and the moments of X_k up to order $p-1$ participate in the definition of g_{p-1}. In particular, assuming that $\mathbb{E}X_k = 0$ for all $k \leq n$ and $\mathbb{E}X_k^2 = \sigma_k^2$, we have

$$\gamma_3 = \sum_{k=1}^n \mathbb{E}X_k^3, \quad \gamma_4 = \sum_{k=1}^n \big(\mathbb{E}X_k^4 - 3\sigma_k^4\big). \tag{10.2}$$

The first two expansions in (10.1) corresponding to $p = 4$ and $p = 5$ are given by

$$g_3(t) = e^{-t^2/2}\Big(1 + \gamma_3 \, \frac{(it)^3}{3!}\Big) \tag{10.3}$$

and

$$g_4(t) = e^{-t^2/2}\Big(1 + \gamma_3 \, \frac{(it)^3}{3!} + \gamma_4 \, \frac{(it)^4}{4!} + \gamma_3^2 \, \frac{(it)^6}{2!\,3!^2}\Big). \tag{10.4}$$

The function g_{p-1} represents the Fourier–Stieltjes transform of a certain signed Borel measure μ_{p-1} on the real line, that is,

$$g_{p-1}(t) = \int_{-\infty}^{\infty} e^{itx} \, d\mu_{p-1}(x), \quad t \in \mathbb{R}.$$

This measure is called the Chebyshev–Edgeworth approximation of order $p-1$ for the distribution of the sum $S_n = X_1 + \dots + X_n$ (or an Edgeworth correction of the normal law). It has a total mass one, and moreover, the moments of S_n and μ_{p-1} coincide up to order $p-1$.

Denote by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2}, \quad x \in \mathbb{R},$$

the standard normal density on the real line, and by

$$H_k(x) = (-1)^k \, (e^{-x^2/2})^{(k)} \, e^{x^2/2}, \qquad k = 0, 1, 2, \dots$$

the Chebyshev-Hermite polynomial of degree k. In particular, $H_1(x) = x$,

$$\begin{aligned} H_2(x) &= x^2 - 1, \qquad & H_4(x) &= x^4 - 6x^2 + 3, \\ H_3(x) &= x^3 - 3x, \qquad & H_5(x) &= x^5 - 10x^3 + 15x. \end{aligned}$$

From (10.1) it follows that μ_{p-1} has density

$$\varphi_{p-1}(x) = \varphi(x) + \varphi(x) \sum \frac{1}{k_1! \dots k_{p-3}!} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_{p-1}}{(p-1)!}\Big)^{k_{p-3}} H_k(x) \tag{10.5}$$

with summation as in (10.1). The corresponding "distribution function" is given by

$$\Phi_{p-1}(x) \, = \, \mu_{p-1}((-\infty, x]) \, = \, \Phi(x) - \varphi(x) \, Q_{p-1}(x), \quad x \in \mathbb{R},$$

where

$$Q_{p-1}(x) = \sum \frac{1}{k_1! \dots k_{p-3}!} \Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_{p-1}}{(p-1)!}\Big)^{k_{p-3}} H_{k-1}(x).$$

It is a polynomial of degree at most $3(p-3)-1$. For the first values, similarly to (10.3)–(10.4) we have

$$Q_3(x) = \frac{\gamma_3}{3!} H_2(x),$$
$$Q_4(x) = \frac{\gamma_3}{3!} H_2(x) + \frac{\gamma_4}{4!} H_3(x) + \frac{\gamma_3^2}{2!\, 3!^2} H_5(x).$$

If $\gamma_3 = 0$ (for example, when the distributions of all X_k are symmetric about the origin), we return to the standard normal distribution function $\Phi_3 = \Phi$, while the next approximating function is simplified to

$$\Phi_4(x) = \Phi(x) - \frac{\gamma_4}{4!} H_3(x)\varphi(x).$$

If L_p is small, the measure μ_{p-1} is close to the standard normal law in weak metrics. Indeed, from Bikjalis' inequality

$$|\gamma_r(X_k)| \le (r-1)!\, \mathbb{E}\, |X_k|^r$$

it follows that

$$|\gamma_r| \le (r-1)!\, L_r, \quad 3 \le r \le p, \tag{10.6}$$

and therefore

$$|\gamma_r| \le (r-1)!\, L_p^{\frac{r-2}{p-2}}, \quad 3 \le r \le p-1$$

(cf. [4] and Remark 6.2 on the monotonicity of the Lyapunov coefficients). Hence, writing

$$k = d + 2\,(k_1 + k_2 + \dots + k_{p-3}), \quad d = k_1 + 2k_2 + \dots + (p-3)\, k_{p-3},$$

we have

$$\Big|\Big(\frac{\gamma_3}{3!}\Big)^{k_1} \dots \Big(\frac{\gamma_{p-1}}{(p-1)!}\Big)^{k_{p-3}}\Big| \le \Big(\frac{L_3}{3}\Big)^{k_1} \dots \Big(\frac{L_{p-1}}{(p-1)}\Big)^{k_{p-3}}$$
$$\le \frac{L_p^{\frac{d}{p-2}}}{3^{k_1} \dots (p-1)^{k_{p-3}}}.$$

Since $3 \le k \le 3(p-3)$, $1 \le d \le p-3$, and using the elementary bound

$$\sum \frac{1}{k_1! \dots k_{p-3}!} \, \frac{1}{3^{k_1} \dots (p-1)^{k_{p-3}}} < e^{1/3} \dots e^{1/(p-1)} < p-1,$$

from (10.1) we get

$$|g_{p-1}(t) - g(t)| \le (p-1) \, \max\Big\{L_p^{\frac{1}{p-2}}, L_p^{\frac{p-3}{p-2}}\Big\} \, \max\{1, |t|^{3(p-3)}\} \, e^{-t^2/2}. \tag{10.7}$$

By a similar argument, from (10.5) we get

$$|\varphi_{p-1}(x) - \varphi(x)| \le c_p \, \max\Big\{L_p^{\frac{1}{p-2}}, L_p^{\frac{p-3}{p-2}}\Big\} \, \max\{1, |x|^{3(p-3)}\} \, \varphi(x). \tag{10.8}$$

Here, the right-hand side is uniformly small over all $x \in \mathbb{R}$ as long as L_p is small. As a consequence, we also have a similar bound on the total variation distance between μ_{p-1} and the standard Gaussian measure,

$$\int_{-\infty}^{\infty} |\varphi_{p-1}(x) - \varphi(x)| \, dx \le c_p \, \max\Big\{L_p^{\frac{1}{p-2}}, L_p^{\frac{p-3}{p-2}}\Big\}. \tag{10.9}$$

We refer the interested reader to [4] for more details on this subject.

11 Generalization of Theorem 2.2

The importance of Edgeworth corrections is explained by the following standard result, cf. e.g. [4]. As before, the independent random variables X_k have mean zero and variances σ_k^2 such that $\sigma_1^2 + \dots + \sigma_n^2 = 1$. We use notations and remarks from the previous section.

Lemma 11.1 *If $L_p < \infty$ for an integer $p \ge 4$, then the characteristic function $f_n(t)$ of the sum $S_n = X_1 + \dots + X_n$ satisfies*

$$|f_n(t) - g_{p-1}(t)| \le c_p L_p \min(1, |t|^p) \, e^{-t^2/8}, \quad |t| \le \frac{1}{L_3}, \tag{11.1}$$

up to some constant $c_p > 0$ depending on p only.

One can now give a more general version of the first claim in Theorem 2.2.

Theorem 11.2 *Suppose that the random variables X_k have finite p-th absolute moments for an integer $p \ge 4$. Then, for any $\delta \in (0, 1]$,*

$$\sup_x |F_n(x) - \Phi_{p-1}(x)| \le c_p \big(\delta^{-\frac{p-2}{2}} \, L_p + L_{2+\delta}^{1/\delta}\big), \tag{11.2}$$

where the constant $c_p > 0$ *depends on* p *only.*

Theorem 2.2 corresponds to (11.2) with $p = 4$ under an additional assumption $\mathbb{E}X_k^3 = 0$ for all $k \le n$, which implies that $\Phi_3 = \Phi$.

Proof If $L_p > 1$, the inequality (11.2) is fulfilled automatically. In this case, the maximum in (10.9) does not exceed a multiple of L_p. Hence, applying this inequality, one may bound the left-hand side of (11.2) by

$$\sup_x |F_n(x) - \Phi(x)| + \sup_x |\Phi_{p-1}(x) - \Phi(x)| \le 1 + \int_{-\infty}^{\infty} |\varphi_{p-1}(x) - \varphi(x)|\, dx \le c_p L_p.$$

On the other hand, the right-hand side of (11.2) is greater than a multiple of L_p.

Now assume that $L_p \le 1$, so that also $L_3 \le L_p^{\frac{1}{p-2}} \le 1$ (cf. Remark 6.2). Put $T_0 = 1/L_3$ and apply the Berry–Esseen inequality (3.1) with

$$f(t) = f_n(t), \quad g(t) = g_{p-1}(t) \quad \text{and} \quad T = L_{2+\delta}^{-1/\delta}.$$

Then, the supremum in (11.2) can be bounded from above by a multiple of

$$\int_{|t|\le T_0} \left|\frac{f_n(t) - g_{p-1}(t)}{t}\right| dt + \int_{T_0<|t|<T} \left|\frac{f_n(t|}{t}\right| dt + \int_{|t|\ge T_0} \left|\frac{g_{p-1}(t)}{t}\right| dt + \frac{A}{T} \tag{11.3}$$

with $A = \|\Phi_{p-1}\|_{\mathrm{Lip}}$. Here, the first integral does not exceed $c_p L_p$, according to (11.1).

Applying the inequality (8.3), we also see that the second integral does not exceed

$$2\int_{T_0}^{\infty} \frac{e^{-\delta t^2/6}}{t}\, dt \le c\, e^{-\delta T_0^2/6} = c\, e^{-\delta/(6L_3^2)}. \tag{11.4}$$

Since $L_3^2 \le L_p^{\frac{2}{p-2}}$ and $x^{\frac{p-2}{2}} e^{-x} \le c_p$ $(x > 0)$, the last expression in (11.4) can be bounded by $c_p L_p\, \delta^{-\frac{p-2}{2}}$ up to some constant $c_p > 0$ depending on p only.

In order to bound the third integral, note that, by (10.7), $|g_{p-1}(t)| \le c_p\, e^{-t^2/4}$ for all $t \in \mathbb{R}$. Hence, this integral does not exceed

$$2\int_{T_0}^{\infty} \frac{e^{-t^2/4}}{t}\, dt \le c\, e^{-1/(4L_3^2)} \le c_p L_3^{\frac{p-2}{2}} \le c_p L_p.$$

Finally, the Lipschitz semi-norm $A = \sup_x |\varphi_{p-1}(x)|$ of Φ_{p-1} in (11.3) is bounded by a p-dependent constant, according to (10.8). Hence, the whole expression in (11.3) is bounded by the right-hand side of (11.2). □

Example 11.3 Given a positive parameter $q \in (\frac{1}{3}, \frac{1}{2})$, let us return to the weighted sums

$$S_n = \frac{1}{b_n} \sum_{k=1}^{n} \frac{1}{k^q}\, \xi_k, \quad b_n = \Big(\sum_{k=1}^{n} \frac{1}{k^{2q}}\Big)^{1/2} \sim n^{\frac{1}{2}-q},$$

assuming that ξ_k are i.i.d. random variables with mean zero, variance one, and with finite moment $\beta_p = \mathbb{E}\,|\xi_1|^p$ of an integer order $p \geq 4$. The Berry–Esseen bound (1.2) gives

$$\sup_x |F_n(x) - \Phi(x)| \leq c_q \beta_3\, \frac{1}{n^{3\,(\frac{1}{2}-q)}},$$

where the constant c_q depends on q only. Here, the right-hand side is worse than the standard rate. This bound may be improved by virtue of Theorem 11.2, by replacing the standard normal distribution function with a suitable Chebyshev–Edgeworth correction. Using any fixed value $\delta \in (0, \frac{1}{q} - 2)$, we have $L_{2+\delta}^{1/\delta} \sim \frac{1}{\sqrt{n}}$, while $L_p \sim n^{-p\,(\frac{1}{2}-q)}$ has a better decay for $p \geq \frac{1}{1-2q}$. Hence, by (11.2),

$$\sup_x |F_n(x) - \Phi_{p-1}(x)| \leq c_{p,q} \beta_p\, \frac{1}{\sqrt{n}}, \quad p \geq \frac{1}{1-2q}.$$

12 Lower Bounds (Proof of Theorem 2.1)

Lemma 11.1 can also be used to derive the lower bounds in Theorem 2.1. In addition, we need the following general relation derived in [3].

Lemma 12.1 *Let U be a function of bounded total variation on the real line with $U(-\infty) = U(\infty) = 0$. For any $T > 0$, we have*

$$\sup_x |U(x)| \geq \frac{1}{3T}\, \bigg|\int_0^T u(t)\, \Big(1 - \frac{t}{T}\Big)\, dt\bigg|,$$

where

$$u(t) = \int_{-\infty}^{\infty} e^{itx}\, dU(x), \quad t \in \mathbb{R},$$

is the Fourier–Stieltjes transform of U.

Proof of Theorem 2.1. Put $g(t) = e^{-t^2/2}$. Being applied to the function $U(x) = F_n(x) - \Phi(x)$ with its Fourier–Stieltjes transform $u(t) = f_n(t) - g(t)$, Lemma 12.1 leads to

$$\Delta_n \geq \frac{1}{3T}\, \bigg|\int_0^T (f_n(t) - g(t))\, \Big(1 - \frac{t}{T}\Big)\, dt\bigg|. \tag{12.1}$$

To further bound from below the integral on the right-hand side, we use the approximation of the characteristic function $f_n(t)$ by the Fourier–Stieltjes transforms $g_3(t)$ and $g_4(t)$ of the Chebyshev–Erdgeworth corrections μ_3 and μ_4 in parts a) and b), respectively.

First, by the triangle inequality, from (12.1) we get, for any $T > 0$,

$$\Delta_n \geq \frac{1}{3T} \left| \int_0^T (g_3(t) - g(t)) \left(1 - \frac{t}{T}\right) dt \right| - \frac{1}{3T} \left| \int_0^T (f_n(t) - g_3(t)) \left(1 - \frac{t}{T}\right) dt \right|. \tag{12.2}$$

Assuming that $T \leq \min(1, 1/L_3)$ and choosing $p = 4$ in (11.1), Lemma 11.1 yields

$$|f_n(t) - g_3(t)| \leq c L_4 \, t^4 e^{-t^2/8}, \quad |t| \leq T.$$

Hence, the second term in (12.2) does not exceed

$$\frac{cL_4}{3T} \int_0^T t^4 e^{-t^2/8} \left(1 - \frac{t}{T}\right) dt \leq cL_4 T^4.$$

Furthermore, according to (10.3),

$$g_3(t) - g(t) = \gamma_3 \, e^{-t^2/2} \, \frac{(it)^3}{3!}.$$

Since $T \leq 1$, the first term in (12.2) is greater than or equal to $c\,|\gamma_3| T^3$. Hence

$$\frac{1}{cT^3} \, \Delta_n \geq |\gamma_3| - c\, L_4 T, \quad 0 < T \leq \min(1, 1/L_3). \tag{12.3}$$

Similarly, for part b) of the theorem, write

$$\Delta_n \geq \frac{1}{3T} \left| \int_0^T (g_4(t) - g(t)) \left(1 - \frac{t}{T}\right) dt \right| - \frac{1}{3T} \left| \int_0^T (f_n(t) - g_4(t)) \left(1 - \frac{t}{T}\right) dt \right|. \tag{12.4}$$

For the value $p = 5$, the bound (11.1) yields

$$|f_n(t) - g_4(t)| \leq c L_5 \, t^5 e^{-t^2/8}, \quad |t| \leq T,$$

where $T \leq \min(1, 1/L_3)$. In this case, the second term in (12.4) does not exceed

$$\frac{cL_5}{3T} \int_0^T t^5 e^{-t^2/8} \left(1 - \frac{t}{T}\right) dt \leq cL_5 T^5.$$

In order to bound from below the first term in (12.4), note that, according to (10.4),

$$\mathrm{Re}(g_4(t) - g(t)) = e^{-t^2/2} \left(\gamma_4 \frac{t^4}{4!} - \gamma_3^2 \frac{t^6}{2!\,3!^2} \right).$$

Hence

$$\left| \int_0^T (g_4(t) - g(t)) \left(1 - \frac{t}{T}\right) dt \right| \geq c_1 \, |\gamma_4| \, T^5 - c_2 \, \gamma_3^2 \, T^7.$$

Here, $\gamma_3^2 \leq 4L_3^2 \leq 4L_4$, by the cumulant inequality (10.6) with $r = 3$. Hence, the first term in (12.4) is greater than or equal to $c_1 \, |\gamma_4| \, T^4 - c_2 \, L_4 \, T^6$. Thus,

$$\frac{1}{cT^4} \, \Delta_n \geq |\gamma_4| - c \, (L_5 T + L_4 T^2), \quad T \leq \min(1, 1/L_4^{1/2}), \tag{12.5}$$

where we strengthened the assumption on T by using $L_3 \leq L_4^{1/2}$.

One can now specialize the relations (12.3) and (12.5) to the scheme of the weighted sums

$$S_n = a_1 \xi_1 + \cdots + a_n \xi_n, \quad a_1^2 + \cdots + a_n^2 = 1,$$

where $(\xi_k)_{1 \leq k \leq n}$ are i.i.d. random variables with mean zero and variance one, assuming that the coefficients a_k are non-negative in part a). Putting

$$\ell_p = \sum_{k=1}^n |a_k|^p, \quad \beta_p = \mathbb{E}\, |\xi_1|^p, \;\; \alpha_3 = \mathbb{E}\xi_1^3,$$

we then have

$$L_p = \beta_p \ell_p, \quad \gamma_3 = \alpha_3 \ell_3, \quad \gamma_4 = (\beta_4 - 3)\, \ell_4.$$

Note also that $L_3 \leq \beta_3$ and $\beta_3 \geq 1$, so that $\min(1, 1/L_3) \geq 1/\beta_3$. Hence, in part a), using $\ell_4 \leq \ell_3$, (12.3) yields

$$\begin{aligned} \frac{1}{cT^3} \, \Delta_n &\geq |\alpha_3| \, \ell_3 - c \beta_4 \ell_4 T \\ &\geq \ell_3 \left(|\alpha_3| - c\beta_4 T \right) \; = \; \frac{L_3}{\beta_3} \left(|\alpha_3| - c\beta_4 T \right), \quad 0 < T \leq \frac{1}{\beta_3}. \end{aligned}$$

Choosing $T = |\alpha_3|/(2c\beta_4)$, we arrive at the required lower bound in (2.2).

For part b), using $\ell_5 \leq \ell_4$ and $L_4 \leq \beta_4$, $\beta_3^2 \leq \beta_4$, (12.5) implies that

$$\frac{1}{cT^4} \, \Delta_n \geq \ell_4 \left(|\beta_4 - 3| \, - c\beta_5 T - c\beta_4 T^2 \right), \quad T \leq \frac{1}{\beta_4^{1/2}}, \tag{12.6}$$

For a sufficiently small value of $T = T(\beta_4, \beta_5)$, the expression in the brackets is larger than $c\beta_4$ with a constant $c > 0$ depending on β_4 and β_5, only, and then the right-hand side dominates a corresponding multiple of L_4. Hence (12.6) leads to the lower bound in (2.3). □

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